Computing Curves on Surfaces

Mckenzie West
University of Wisconsin-Eau Claire

Jen Berg
Bucknell University

Rachel Davis
University of Wisconsin-Madison

Marie Jameson
University of Tennessee

Bianca Thompson
Westminster College

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Del Pezzo surfaces are rational surfaces that exist in degrees 1 thru 9

(degree $\geq 5$) satisfy the Hasse Principle

$$S(\mathbb{Q}) = \emptyset \iff S(\mathbb{Q}_p) = \emptyset \text{ for some } p, \text{ prime or } \infty.$$ 

(degree 4) smooth intersections of two quadrics in $\mathbb{P}^4$
(degree 3) cubic surfaces in $\mathbb{P}^3$
(degree 2) $w^2 = f_4(x, y, z, w)$ in $\mathbb{P}(1, 1, 1, 2)$
(degree 1) defined by a degree 6 polynomial in $\mathbb{P}(1, 1, 2, 3)$
e.g., $w^2 = z^3 + 27x^6 + 16y^6$
If $S$ is a del Pezzo surface of degree $d$ defined over $\mathbb{Q}$, then $S$ contains EXACTLY the given number of exceptional curves:

<table>
<thead>
<tr>
<th>$d$</th>
<th>#ExC’s</th>
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<tbody>
<tr>
<td>9</td>
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<td>56</td>
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Curves on del Pezzo Surfaces

We can use these curves to generate nearly all of Pic $\overline{S} \simeq \mathbb{Z}^r$, a group of equivalence classes of curves on $\overline{S}$.

<table>
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<tr>
<th>$d$</th>
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Example.
Consider the cubic surface:

\[ S : x^3 + y^3 + z^3 = w^3 \]

The lines are easy to find.
Let Me Show You

Example.

We isolate pairs of variables and equate as needed:

\[ x^3 + y^3 + z^3 = w^3 \]

\[ \begin{cases} 
  x = \zeta_3^i w \\
  y = -\zeta_3^j z 
\end{cases} \quad \text{for } 0 \leq i, j \leq 2 \]

And that makes 27 lines with field of definition \( Q(\zeta_3^3) \).

Oh how lucky we were with this equation.
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for \(0 \leq i, j \leq 2\)

And that makes 27 lines with field of definition \(\mathbb{Q}(\zeta_3)\). Oh how lucky we were with this equation.
Another Cubic

Example.
Now consider the cubic surface

\[ S : x^3 + 2xy^2 + 11y^3 + 3xz^2 + 5y^2w + 7zw^2 \]

the lines here are not so obvious.
Let’s use a computer to find the lines and their field of definition.

Linear Algebra.
All lines in three dimensional projective space can be parametrized as

\[ \begin{bmatrix} x & y & z & w \end{bmatrix} = \begin{bmatrix} s & t \end{bmatrix} A \]

where \( A \) is a \( 2 \times 4 \) matrix in reduced row echelon form.
Example.

\[ S : x^3 + 2xy^2 + 11y^3 + 3xz^2 + 5y^2w + 7zw^2 \]

\[ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} s \\ t \end{bmatrix} A \]

For example, let \( A = \begin{bmatrix} 1 & 0 & \alpha & \beta \\ 0 & 1 & \gamma & \delta \end{bmatrix} \).

Substitute into the polynomial for \( S \) via the parametrization,

\[ s^3 + 2st^2 + 11t^3 + 3s(\alpha s + \gamma t)^2 + 5t^2(\beta s + \delta t) + 7(\alpha s + \gamma t)(\beta s + \delta t)^2 \]
Approximate Algorithm.

Repeat the following for all possible matrices, A:

- Isolate coefficients of $s^3$, $s^2t$, $st^2$ and $t^3$.

\[
\begin{align*}
7ab^2 + 3a^2 + 1 \\
7b^2c + 14abd + 6ac \\
14bcd + 7ad^2 + 3c^2 + 5b + 2 \\
7cd^2 + 5d + 11
\end{align*}
\]
Our Sage Work

**Approximate Algorithm.**

Repeat the following for all possible matrices, $A$:

- Isolate coefficients of $s^3$, $s^2 t$, $st^2$ and $t^3$.
- Form an ideal in $\mathbb{Q}[\alpha, \beta, \gamma, \delta]$. 
Approximate Algorithm.

Repeat the following for all possible matrices, $A$:

- Isolate coefficients of $s^3$, $s^2 t$, $st^2$ and $t^3$.
- Form an ideal in $\mathbb{Q}[\alpha, \beta, \gamma, \delta]$.
- Use the Sage Groebner basis functionality to obtain a polynomial defining a number field over which the solutions to the system exist.
Result (Berg, Davis, Jameson, Thompson, W.).

Input: Homogeneous polynomial defining a nice cubic surface
Output: Polynomial \( f \) whose splitting field \( L \) is the field of definition of the 27 lines on the surface.
Result (Berg, Davis, Jameson, Thompson, W.).

Input: Homogeneous polynomial defining a nice cubic surface
Output: Polynomial $f$ whose splitting field $L$ is the field of definition of the 27 lines on the surface.

Example.

```
get_polynomial_to_split(f)
```

\[
T^{27} + \frac{99}{5} T^{26} + \frac{3299}{25} T^{25} + \frac{36289}{125} T^{24} + \frac{264}{49} T^{23} + \frac{10296}{245} T^{22} + \frac{7725912}{60025} T^{21} + \frac{10026984}{42875} T^{20} + \frac{51920262}{117649} T^{19} \\
+ \frac{605313522}{588245} T^{18} + \frac{4659518538}{2941225} T^{17} + \frac{11953541358}{2941225} T^{16} + \frac{69318152838}{5764801} T^{15} + \frac{626176481634}{28824005} T^{14} \\
+ \frac{156330541343898}{7061881225} T^{13} + \frac{70137153565182}{7061881225} T^{12} + \frac{6440174126145}{13841287201} T^{11} + \frac{44233686867843}{69206436005} T^{10} \\
+ \frac{33749245501389}{49433168575} T^9 + \frac{250779458513133}{346032180025} T^8 + \frac{10050474124746}{13841287201} T^7 + \frac{4683175938738}{9886633715} T^6 \\
+ \frac{990986421546}{49433168575} T^5 - \frac{6392766446550}{13841287201} T^4 - \frac{11720071818675}{13841287201} T^3 - \frac{1718943866739}{1977326743} T^2 \\
- \frac{170175442807161}{346032180025} T - \frac{207992207875419}{1730160900125}
\]
Second Algorithm

**Result (Berg, Davis, Jameson, Thompson, W.).**

Input: Homogeneous polynomial defining a nice cubic surface
Output: Parametrizations of the 27 lines on the surface defined by the polynomial.

Warning: This algorithm only works for rather simple surfaces.

Example.

```python
P.<x,y,z,w> = PolynomialRing(QQ)
find_27_lines(xˆ3 + yˆ3 + zˆ3 - wˆ3)
```

\[
\begin{align*}
y + (-\alpha + 1)w, & \quad x + z \\
\alpha w, & \quad x + z \\
y - w, & \quad x + z \\
y + z, & \quad x + (-\alpha + 1)w \\
\end{align*}
\]

...
Second Algorithm

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[y + (-\alpha + 1) w, x + z]
[y + \alpha w, x + z]
[y - w, x + z]
[y + z, x + (-\alpha + 1) w]
::
A Moment of Reflection

Magma

R1<x,y,z,w> := PolynomialRing(Rationals(),4);
f := x^3+2*x*y^2+11*y^3+3*x*z^2+5*y^2*w+7*z*w^2;
S := Scheme(ProjectiveSpace(R1),f);

R2<a,b,c,d> := PolynomialRing(Rationals(),4);
R3<s,t> := PolynomialRing(R2,2);
g := Evaluate(f,[s,t,a*s+c*t,b*s+d*t]);
I:=Ideal(Coefficients(g));
G:=GroebnerBasis(I);

X := Scheme(AffineSpace(R2),Coefficients(g));
pts, K := PointsOverSplittingField(X);
A Little Motivation

Let $S$ be a *nice* surface.

$$S(\mathbb{Q}) \quad \Pi'_p S(\mathbb{Q}_p) = S(\mathbb{A})$$

↑

Cool! \quad OK

Hard! \quad Easy

Sad news:
– It is not necessarily guaranteed that $S(\mathbb{Q})$ is dense in $S(\mathbb{A})$.
– It is not even guaranteed that if $S(\mathbb{A}) \neq \emptyset$, then $S(\mathbb{Q}) \neq \emptyset$.
– It is incredibly difficult, in general, to learn about $S(\mathbb{Q})$, if we know $S(\mathbb{A})$. 
Let $S$ be a *nice* surface.

$$S(\mathbb{Q}) \subseteq \prod'_p S(\mathbb{Q}_p) = S(\mathbb{A})$$

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Let $S$ be a *nice* surface.

\[
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\[
\uparrow \quad \uparrow
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Let $S$ be a *nice* surface defined over $\mathbb{Q}$.

$$S(\mathbb{Q}) \subseteq \prod'_p S(\mathbb{Q}_p) = S(\mathbb{A})$$

Cool! \hspace{1cm} OK

Hard! \hspace{1cm} Easy

Good news: Evidence shows $S(\mathbb{A}) \subseteq \text{Br}$ if and only if $S(\mathbb{Q}) = \emptyset$ for rational surfaces.

The Brauer–Manin Obstruction is the only obstruction to the Hasse Principle – When computing $S(\mathbb{A})$, we are really looking for $\text{Br}_S/\text{Br}_\mathbb{Q}$, which can be computed using the curves on the surface.
A Little Motivation

Let $S$ be a *nice* surface defined over $\mathbb{Q}$.

$$S(\mathbb{Q}) \subseteq S(\mathbb{A})^{Br} \subseteq \prod_p S(\mathbb{Q}_p) = S(\mathbb{A})$$

Cool!  OK
Hard!  Easy

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The Brauer–Manin Obstruction is the only obstruction to the Hasse Principle – When computing $S(\mathbb{A})^{Br}$ we are really looking for $Br_{S/\mathbb{Q}}$, which can be computed using the curves on the surface.
A Little Motivation

Let $S$ be a \textit{nice} surface defined over $\mathbb{Q}$.

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S(\mathbb{Q}) \subseteq S(A)^{Br} \subseteq \prod_p S(\mathbb{Q}_p) = S(A)
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  *The Brauer–Manin Obstruction is the only obstruction to the Hasse Principle*

- When computing $S(\mathbb{A})^{Br}$ we are really looking for $Br S / Br \mathbb{Q}$, which can be computed using the *curves on the surface*. 
Our Ultimate Long-Term Goal

**Theorem (Corn).**

If $S$ is a del Pezzo surface of degree $d$, then $\text{Br } S / \text{Br } \mathbb{Q}$ is isomorphic to one of the following:

- all $d$: \{1\}
- if $d \leq 4$: $\mathbb{Z}/2\mathbb{Z}$, $(\mathbb{Z}/2\mathbb{Z})^2$
- if $d \leq 3$: $\mathbb{Z}/3\mathbb{Z}$, $(\mathbb{Z}/3\mathbb{Z})^2$
- if $d \leq 2$: $(\mathbb{Z}/2\mathbb{Z})^3$, $(\mathbb{Z}/2\mathbb{Z})^4$, $(\mathbb{Z}/2\mathbb{Z})^5$, $(\mathbb{Z}/2\mathbb{Z})^6$, $\mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^2$, $(\mathbb{Z}/4\mathbb{Z})^2$
- if $d \leq 1$: 14 element list that includes $\mathbb{Z}/5\mathbb{Z}$ and $\mathbb{Z}/6\mathbb{Z}$

**Goal.**

Find a 5-torsion element in $\text{Br } S / \text{Br } \mathbb{Q}$. 
An Important Result to Consider

**Theorem (Carter).**

Let $S$ be a del Pezzo surface of degree 1 defined over $\mathbb{Q}$. There is a nontrivial element of $(\text{Br } S/\text{Br } \mathbb{Q})[5]$ if and only if

there is a “ten-tuple-five” on $\overline{S}$ that is stabilized by the absolute Galois group $G = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that $G$ does not fix any

(a) “five” within the “ten-tuple-five”;
(b) “quadruple-five” within the “ten-tuple-five”;
(c) “ten-tuple-one” within the “ten-tuple-five”;
(d) “ten-tuple-three” within the “ten-tuple-five”.