The arithmetic of a family of degree-two *semi*-diagonal K3 surfaces

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Rational Points

\textbf{Question}

\textbf{Fact}

\[ X(Q) \subseteq X(A) \]

\[ B_r \subseteq X(A) \]

\textbf{Good News:}

'Is \[ X(A) = \emptyset \]?' is easy!
Question

What is $X(\mathbb{Q})$?
Question

Is $X(\mathbb{Q}) \neq \emptyset$?
Question

Is $X(\mathbb{Q}) = \emptyset$?
Rational Points

Question

Is $X(\mathbb{Q}) = \emptyset$?

Fact

$X(\mathbb{Q}) \subseteq X(\mathbb{A})^{Br} \subseteq X(\mathbb{A})$
Rational Points

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Is $X(Q) = \emptyset$?

Fact

$X(Q) \subseteq X(A)^{Br} \subseteq X(A)$

**Good News:** ’Is $X(A) = \emptyset$?’ is easy!
A little about $X(\mathbb{A})^{Br}$
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- $\text{Br}_1(X)/\text{Br}(\mathbb{Q}) \simeq H^1(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \text{Pic}(\overline{X}))$ which is a finite group
- For every $\alpha \in \text{Br}_1(X)$,

  $X(\mathbb{Q}) \subseteq X(\mathbf{A})^{Br} \subseteq X(\mathbf{A})^{Br_1} \subseteq X(\mathbf{A})^{\alpha}$. 

Geometry of K3 surfaces/\mathbb{Q}
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- \exists 1 \leq \rho(X) \leq 20 and \( C_1 \ldots C_{\rho(X)} \) divisors (curves) on \( \overline{X} \) such that

\[
\text{Pic}(\overline{X}) \cong \text{NS}(\overline{X}) \cong \mathbb{Z}^{\rho(X)} \cong \langle C_1, \ldots, C_{\rho(X)} \rangle
\]
Theorem 1

Define $X_D: w^2 = x^6 + y^6 + z^6 + D(xyz)^2$. Then for a generic $D \in \mathbb{Q}$,

$$\text{Pic}(X_D) \cong \mathbb{Z}_{19}.$$ 

Otherwise, $\rho(X_D) = 20$. 

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Theorem 1 (Bouyer, Costa, Festi, Nicholls, W.)

Define \( X_D : w^2 = x^6 + y^6 + z^6 + D(xyz)^2 \). Then for a generic \( D \in \mathbb{Q} \),

\[
\text{Pic}(\overline{X}_D) \simeq \mathbb{Z}^{19} \quad (\text{i.e.,} \quad \rho(X_D) = 19).
\]

Otherwise, \( \rho(X_D) = 20 \).
Theorem 2 (Bouyer, Costa, Festi, Nicholls, W.)

For a generic $D$, $B_r^1(X, D) / B_r(Q) \cong (\mathbb{Z}/2\mathbb{Z})^3$. 
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For a generic $D$, 

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- Such a map exists from \( X_D \) to \( \mathcal{E} \), where \( \mathcal{E} \) is the elliptic fibration

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  \]
- $\rho(\mathcal{E}) = 19$ because $\mathcal{E}$ has
  - two $E_8$ fibers, $t = 0$ and $t = \infty$
  - and one $A_1$ fiber, $t = -1$
The projection \( x:y:z:w \mapsto x:y:z \) from \( X_D \) to \( P^2 \) is a cover ramified along the sextic \( C \):
\[ x^6 + y^6 + z^6 + D(xyz)^2. \]
Write down eight conics which intersect \( C \) tangentially at six points each. Let \( C_1, \ldots, C_8 \) be their preimages as divisors on \( X_D \).

Take \( G = \text{Aut}(X_D) \).

Define \( \Lambda := G \cdot \langle C_1, \ldots, C_8 \rangle \subseteq \text{Pic}(X_D). \)

Lemma 3 (Bouyer, Costa, Festi, Nicholls, W.)
The projection \([x : y : z : w] \mapsto [x : y : z]\) from \(X_D\) to \(\mathbb{P}^2\) is a cover ramified along the sextic \(C : x^6 + y^6 + z^6 + D(xyz)^2\).
The projection $[x : y : z : w] \mapsto [x : y : z]$ from $X_D$ to $\mathbb{P}^2$ is a cover ramified along the sextic $C : x^6 + y^6 + z^6 + D(xyz)^2$.

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**Lemma 3 (Bouyer, Costa, Festi, Nicholls, W.)**

\[ \Lambda = \text{Pic}(\overline{X}_D) \]
$H^1(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \text{Pic}(\overline{X_D}))$

If $K$ is the field of definition for $C_1,...,C_8$, then generically $[K:Q] = 96 = 3 \cdot 2^5$.

$\text{Gal}(K/Q) \cong D_4 \times C_2 \times S_3$.

$H^1(\text{Gal}(\overline{Q}/Q), \text{Pic}(X_D)) \cong H^1(\text{Gal}(K/Q), \text{Pic}(X_D \times \mathbb{Q}_K))$.

Write $\text{Pic}(X_D)$ and $\sigma \in \text{Gal}(K/Q)$ as matrices.

By MAGMA:

$\text{Br}^1(X_D)/\text{Br}(Q) \cong H^1(\text{Gal}(K/Q), \text{Pic}(X_D \times \mathbb{Q}_K) \cong (\mathbb{Z}/2\mathbb{Z})^3$. 
\[ H^1(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \text{Pic}(\overline{X}_D)) \]

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- $H^1(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \text{Pic}(\overline{X_D})) \cong H^1(\text{Gal}(K/\mathbb{Q}), \text{Pic}(X_D \times_{\mathbb{Q}} K))$
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Paths for Future Work
Proposition

There is an isomorphism

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\frac{(\text{Pic}(X_D)/m \text{Pic}(X_D))^{\text{Gal}(\overline{Q}/Q)}}{\text{Pic}(X_D)/2 \text{Pic}(X_D)} \rightarrow H^1(\text{Gal}(\overline{Q}/Q), \text{Pic}(X_D))[2]
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C \mapsto (\sigma \mapsto \frac{1}{2}(\sigma C - C))
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Proposition

Take \(L/\mathbb{Q}\) a cyclic extension, and \(f \in k(X)\). Then \((L/\mathbb{Q}, f) \in \text{Br}(X_D)\) if and only if \(\text{div}(f) = \text{Norm}_{L/\mathbb{Q}}(D)\) for a \(D \in \text{Div}(X_D \times \mathbb{Q} L)\).
Goals

- Write down elements of $\text{Br}_1(X_D)/\text{Br}(\mathbb{Q})$. 
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- Compute $X_D(A)^{\text{Br}_1}$.
- Write down elements of $\text{Br}(X_D)/\text{Br}_1(X_D)$ and compute $X_D(A)^{\text{Br}}$. 

$w_2 = Ax^6 + By^6 + Cz^6 + D(xyz)^2$.
Goals

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- Write down elements of $\text{Br}(X_D)/\text{Br}_1(X_D)$ and compute $X_D(A)^{\text{Br}}$.
- Generalize these results to

$$X_{A,B,C,D} : w^2 = Ax^6 + By^6 + Cz^6 + D(xyz)^2.$$