The arithmetic of a family of degree-two K3 surfaces

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Rational Points

Rationale

Study the family of curves $X_{A,B,C,D}$ of the form $w^2 = Ax^6 + By^6 + Cz^6 + D(xyz)^2$ where $A, B, C, D \in \mathbb{Q}$.

Specifically we ask about $X_{\mathbb{Q}}$. 

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K3 Arithmetic
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K3 surface
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- deg 2: $w^2 = f_6(x, y, z)$ in $\mathbb{P}(1, 1, 1, 3)$
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- deg 2: $w^2 = f_6(x, y, z)$ in $\mathbb{P}(1, 1, 1, 3)$
- deg 4: $f_4(x, y, z, w) = 0$ in $\mathbb{P}^3$
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- **deg 2**: $w^2 = f_6(x, y, z)$ in $\mathbb{P}(1, 1, 1, 3)$
- **deg 4**: $f_4(x, y, z, w) = 0$ in $\mathbb{P}^3$
- **deg 8**: $\{Q_1 \cap Q_2 \cap Q_3\}$ in $\mathbb{P}^5$
The Hasse principle

Recall

\[ X(\mathbb{Q}) \subseteq X(\mathbb{A}) \]

Definition

We say \( X \) satisfies the Hasse principle if

\[ X(\mathbb{Q}) = \emptyset \Rightarrow X(\mathbb{A}) = \emptyset \]

Good news: determining whether \( X(\mathbb{A}) = \emptyset \) is a finite computation.

Bad news: the Hasse principle fails often.

Good news: (Manin)

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\[ X(\mathbb{Q}) \subseteq X(\mathbb{A})^{\text{Br}} \subseteq X(\mathbb{A}) \]
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K3 Arithmetic
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- Define $Br(X) := H^2_{\text{ét}}(X, \mathbb{G}_m)$.
- Elements of $Br(X)$ can be realized as elements of $Br(k(X))$.
- Using this, we can compute $X(A)^{Br} := \bigcap_{\alpha \in Br(X)} X(A)^{\alpha}$. 

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Filtration on $\text{Br}(X)$
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Filtration

$$\text{Br}_0(X) \subseteq \text{Br}_1(X) \subseteq \text{Br}(X)$$

where $\text{Br}_0(X) := \text{im}(\text{Br} \mathbb{Q} \to \text{Br} X)$ and
$\text{Br}_1(X) := \ker(\text{Br}(X) \to \text{Br}(\overline{X}))$
Filtration on $\text{Br}(X)$

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\text{where } \text{Br}_0(X) & := \text{im} (\text{Br} \mathbb{Q} \to \text{Br} X) \text{ and} \\
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\end{align*}

Naturally for every $\alpha \in \text{Br}_1(X),$

\[ X(\mathbb{Q}) \subseteq X(\mathbb{A})^\text{Br} \subseteq X(\mathbb{A})^\text{Br}_1 \subseteq X(\mathbb{A})^\alpha. \]
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Naturally for every $\alpha \in \text{Br}_1(X)$,

\[
X(\mathbb{Q}) \subseteq X(\mathbb{A})^{\text{Br}} \subseteq X(\mathbb{A})^{\text{Br}_1} \subseteq X(\mathbb{A})^\alpha.
\]

Moreover $\text{Br}_1(X)/\text{Br}_0(X) \simeq H^1(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \text{Pic}(\overline{X}))$. 
Brauer groups and K3 surfaces (char 0)
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Filtration

\[ \text{Br}_0(X) \subseteq \text{Br}_1(X) \subseteq \text{Br}(X) \]

\[ \text{Br}_1(X) / \text{Br}_0(X) \cong H^1(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \text{Pic}(\overline{X})) \]

- \( \exists 1 \leq \rho(X) \leq 20 \) and \( C_1 \ldots C_{\rho(X)} \) divisors (curves) on \( \overline{X} \) such that

\[ \text{Pic}(\overline{X}) \cong \text{NS}(\overline{X}) \cong \mathbb{Z}^{\rho(X)} \cong \langle C_1, \ldots, C_{\rho(X)} \rangle \]

Skorobogatov/Zarkin, 2008

\[ \text{Br}(X) / \text{Br}_0(X) \text{ is finite} \]
Brauer groups and K3 surfaces (char 0)

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- \(\text{Br} \overline{X} \cong (\mathbb{Q}/\mathbb{Z})^{22-\rho(X)}\)
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- \( \text{Br} \overline{X} \cong (\mathbb{Q}/\mathbb{Z})^{22-\rho(X)} \)

- (Skorobogatov/Zarkin, 2008) \( \text{Br}(X)/\text{Br}_0(X) \) is finite
Our surfaces
Consider $X_{A,B,C,D}$ defined by

$$w^2 = Ax^6 + By^6 + CZ^6 + D(\text{xyz})^2$$

in $\mathbb{P}(1,1,1,3)$, where $A, B, C, D \in \mathbb{Q}$. 
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Compute $\text{Pic} \overline{X}_{A,B,C,D}$. 
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Goal

Compute $\text{Pic}(\overline{X}_{A,B,C,D})$.

Initial step: Scale $x, y, z$ over $\overline{\mathbb{Q}}$ to eliminate coefficients.
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Consider $X_{A,B,C,D}$ defined by

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Goal

Compute $\text{Pic} \overline{X}_{A,B,C,D}$.

Initial step: Scale $x, y, z$ over $\overline{\mathbb{Q}}$ to eliminate coefficients. Now we have $X_D$ defined by:

$$w^2 = x^6 + y^6 + z^6 + D(xyz)^2,$$

with $D$ in some finite extension of $\mathbb{Q}$. 
Theorem 1

Theorem 1 (Bouyer, Costa, Festi, Nicholls, W.):

Define \( D_w = x^6 + y^6 + z^6 + D(xyz)^2 \). Then for a generic \( D \),

\[
\text{Pic}(X_D) \cong \mathbb{Z}_{19}, \quad \rho(X_D) = 19.
\]

Otherwise, \( \rho(X_D) = 20 \).
Theorem 1

Definition $X_D$: $w^2 = x^6 + y^6 + z^6 + D(xyz)^2$. Then for a generic $D$,

\[ \text{Pic}(X_D) \cong \mathbb{Z}^{19} \] (i.e., $\rho(X_D) = 19$).

Otherwise, $\rho(X_D) = 20$. 

Theorem 1 (Bouyer, Costa, Festi, Nicholls, W.)
Rank 19 K3 surfaces

Note

Any one-dimensional family of K3 surfaces of generic rank 19 is parameterized by a modular curve whose CM points correspond to the rank 20 specializations.
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Any one-dimensional family of K3 surfaces of generic rank 19 is parameterized by a modular curve whose CM points correspond to the rank 20 specializations.
Theorem 2

\[ \text{Theorem 2 (Bouyer, Costa, Festi, Nicholls, W.)} \]

For a generic D, \( Br_1(X_D) / Br_0(X_D) \cong (\mathbb{Z} / 2\mathbb{Z})^3 \).
Theorem 2 (Bouyer, Costa, Festi, Nicholls, W.)

For a generic $D$,

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- \( \rho(X_D) \) is invariant under dominant maps of K3 surfaces
- Such a map exists from \( X_D \) to \( \mathcal{E} \), where \( \mathcal{E} \) is the elliptic fibration

\[ \mathcal{E} : \hat{y}^2 = \hat{x}^3 + Dt^2\hat{x}^2 + t^5(t + 1)^2 \]
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  \[ \mathcal{E} : \hat{y}^2 = \hat{x}^3 + Dt^2\hat{x}^2 + t^5(t + 1)^2 \]
- \( \rho(\mathcal{E}) = 19 \) because \( \mathcal{E} \) has
  - two \( E_8 \) fibers, \( t = 0 \) and \( t = \infty \)
  - and one \( A_1 \) fiber, \( t = -1 \)
The projection \( x:y:z:w \) from \( X_D \) to \( P^2 \) is a cover ramified along the sextic \( C : x^6 + y^6 + z^6 + D(xyz)^2 \).

Write down eight conics which intersect \( C \) tangentially at six points each. Let \( C_1, \ldots, C_8 \) be their preimages as divisors on \( X_D \).

Take \( G = \text{Aut}(X_D) \).

Define \( \Lambda := G \cdot \langle C_1, \ldots, C_8 \rangle \subseteq \text{Pic}(X_D) \).

Lemma 3 (Bouyer, Costa, Festi, Nicholls, W.)

\[ \Lambda = \text{Pic}(X_D) \]
The projection \([x : y : z : w] \mapsto [x : y : z]\) from \(X_D\) to \(\mathbb{P}^2\) is a cover ramified along the sextic \(C : x^6 + y^6 + z^6 + D(xyz)^2\).
The projection $[x : y : z : w] \mapsto [x : y : z]$ from $X_D$ to $\mathbb{P}^2$ is a cover ramified along the sextic $C : x^6 + y^6 + z^6 + D(xyz)^2$.

Write down eight conics which intersect $C$ tangentially at six points each. Let $C_1, \ldots, C_8$ be their preimages as divisors on $X_D$. 
Pic(\(\overline{X}_D\))

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- Write down eight conics which intersect \(C\) tangentially at six points each. Let \(C_1, \ldots, C_8\) be their preimages as divisors on \(X_D\).
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Motivation

Our Results

Proof Methods

Future Work

\[ \text{Pic}(X_D) \]

- The projection \([x : y : z : w] \mapsto [x : y : z]\) from \(X_D\) to \(\mathbb{P}^2\) is a cover ramified along the sextic \(C : x^6 + y^6 + z^6 + D(xyz)^2\).
- Write down eight conics which intersect \(C\) tangentially at six points each. Let \(C_1, \ldots, C_8\) be their preimages as divisors on \(X_D\).
- Take \(G = \text{Aut}(X_D)\).
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The projection $[x : y : z : w] \mapsto [x : y : z]$ from $X_D$ to $\mathbb{P}^2$ is a cover ramified along the sextic $C : x^6 + y^6 + z^6 + D(xyz)^2$.

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**Lemma 3 (Bouyer, Costa, Festi, Nicholls, W.)**

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- \( \text{Gal}(K/\mathbb{Q}) \cong D_4 \times C_2 \times S_3 \)
- \( H^1(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \text{Pic}(\overline{X_D})) \cong H^1(\text{Gal}(K/\mathbb{Q}), \text{Pic}(X_D \times_Q K)) \)
$H^1(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \text{Pic}({\overline{X}_D}))$

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- $H^1(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \text{Pic}({\overline{X}_D})) \cong H^1(\text{Gal}(K/\mathbb{Q}), \text{Pic}(X_D \times_{\mathbb{Q}} K))$

- Write $\text{Pic}({\overline{X}_D})$ and $\sigma \in \text{Gal}(K/\mathbb{Q})$ as matrices.
\[ H^1(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \text{Pic}(\overline{X}_D)) \]

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- Write \( \text{Pic}(\overline{X}_D) \) and \( \sigma \in \text{Gal}(K/\mathbb{Q}) \) as matrices.

- By MAGMA:

\[ \text{Br}_1(X_D)/\text{Br}(\mathbb{Q}) \cong H^1(\text{Gal}(K/\mathbb{Q}), \text{Pic}(X_D \times_K K)) \cong (\mathbb{Z}/2\mathbb{Z})^3. \]
Goals

- Write down elements of $\text{Br}_1(X_D)/\text{Br}(\mathbb{Q})$. 
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- Compute $X_D(A)^{\text{Br}_1}$.
- Write down elements of $\text{Br}(X_D)/\text{Br}_1(X_D)$ and compute $X_D(A)^{\text{Br}}$.  

Generalize these results to $X_A, B, C, D$: $w^2 = Ax^6 + By^6 + Cz^6 + D(xyz)^2$. 

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K3 Arithmetic
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- Write down elements of $\text{Br}(X_D)/\text{Br}_1(X_D)$ and compute $X_D(A)^{\text{Br}}$.
- Generalize these results to

\[ X_{A,B,C,D} : w^2 = Ax^6 + By^6 + Cz^6 + D(xyz)^2. \]