1. Introduction

In this text, we will be studying Lie algebra structures on a vector space $V$ over a field $K$. Most of the time, the field will be $\mathbb{C}$, and the vector space will be finite dimensional, but the general picture is the same over all fields and there are many infinite dimensional Lie algebras of interest to both mathematicians and physicists.

**Definition 1.1.** A Lie algebra on a $K$-vector space $V$ is a binary map $[\cdot, \cdot] : V \times V \to V$, called the (Lie) bracket, satisfying the following:

1. $[u + u', v] = [u, v] + [u', v]$, (Left additivity).
4. $[u, [v, w]] = [[u, v], w] + [v, [u, w]]$, (The Jacobi identity).

**Definition 1.2.** Suppose that $V$ and $W$ are $K$-vector spaces. A map $\beta : V \times V \to W$ is called bilinear if it satisfies the following:

1. $\beta(u + u', v) = \beta(u, v) + \beta(u', v)$, (Left additivity).
2. $\beta(u, v + v') = \beta(u, v) + \beta(u, v')$, (Right additivity).
3. $\beta(cu, v) = c\beta(u, v)$, (Left Homogeneity).
4. $\beta(u, cv) = c\beta(u, v)$, (Right Homogeneity).

If $V$ has the structure of a Lie algebra, then the bracket is bilinear. In fact, it is easy to see that a Lie algebra is given by a bilinear bracket which satisfies antisymmetry and the Jacobi identity. A vector space $A$ equipped with a bilinear map $\beta : A \times A \to A$ is called an algebra. Thus, a Lie algebra is an example of an algebra.

**Definition 1.3.** An **associative algebra over** $K$ is a $K$-vector space $A$ equipped with a bilinear map $m : A \times A \to A$ which satisfies the associative law

$$m(m(a, b), c) = m(a, m(b, c)).$$

If we denote the map $m$ by juxtaposition; i.e. $m(a, b) = ab$, then the associative law takes the familiar form

$$(ab)c = a(bc).$$

It is a reasonable question to ask why the Jacobi identity would arise in any natural fashion. The example below gives some justification.

**Exercise 1.** Let $A$ be an associative algebra over $\mathbb{K}$. Then $A$ is also a Lie algebra over $\mathbb{K}$ if we equip it with the bracket

$$[a, b] = ab - ba.$$
Definition 1.4. If $V$ is a $\mathbb{K}$-vector space, then $\mathfrak{gl}(V)$ is the set of all linear maps from $V$ to $V$. It is an associative algebra under the composition product $\lambda \mu = \lambda \circ \mu$. We also consider $\mathfrak{gl}(V)$ to be a Lie algebra under the associated bracket
$$[\lambda, \mu] = \lambda \mu - \mu \lambda.$$

Definition 1.5. A subalgebra $U$ of a Lie algebra $V$ is a subspace of $V$ which is closed under the bracket operator.

Exercise 2. Let $U$ be a subalgebra of a Lie algebra $V$. Then show that $U$ is a Lie algebra under the bracket operator.

Example 1. A derivation of an algebra $A$ is a linear map $\delta : A \to A$ satisfying the Leibniz Rule
$$\delta(ab) = \delta(a)b + a\delta(b).$$

Exercise 3. Show that the set $\text{Der}(A)$ of all derivations of an algebra $A$ is a Lie algebra under the bracket of linear maps.

Definition 1.6. A morphism of Lie algebras is a linear map $f : V \to W$, where $V$ and $W$ are Lie algebras over $\mathbb{K}$ satisfying
$$f([u,v]) = [f(u), f(v)],$$
where we denote the Lie bracket on both spaces by $[\cdot, \cdot]$.

Definition 1.7. If $V$ is a Lie algebra, then the adjoint operator given by an element $u \in V$, is the map $\text{ad}_u : V \to V$ given by
$$\text{ad}_u(v) = [u, v].$$

Exercise 4. Show that the adjoint operator $\text{ad}_u$ is a derivation of $V$. Notice that the Jacobi identity is precisely the statement that $\text{ad}_u$ is a derivation of the bracket operation!

Exercise 5. Show that the map $V \to \text{Der}(V)$, given by $u \mapsto \text{ad}_u$, is a morphism of Lie algebras.

Definition 1.8. A subspace $U$ of a Lie algebra $V$ is called an ideal or Lie ideal if $[u, v] \in U$, for all $u \in U$, $v \in V$.

Exercise 6. Show that if $f : U \to V$ is a morphism of Lie algebras then the kernel $\ker f$ of $f$ given by
$$\ker f = \{u \in U | f(u) = 0\}$$
is an ideal in $U$.

Also show that the image $\text{Im} f$ of $f$, given by
$$\text{Im} f = \{v \in V | v = f(u) \text{ for some } u \in U\}$$is a subalgebra of $V$.

Exercise 7. Let $\text{ad} : V \to \text{Der}(V)$ be the adjoint map. Show that if $\delta \in \text{Der}(V)$, then $[\delta, \text{ad}_u] = \text{ad}_{\delta(u)}$. Conclude from this that $\text{ad}(V) = \text{Im}(\text{ad})$ is an ideal in $\text{Der}(V)$.
Definition 1.9. If $X$ is any set, the Kronecker delta function on $X \times X$ is given by

$$\delta^A_B = \begin{cases} 1, & \text{if } A = B \\ 0, & \text{otherwise} \end{cases}.$$

Definition 1.10. The map $\psi^{ij}_k : V \times V \to V$ is given by

$$\psi^{ij}_k(e_m, e_n) = (\delta^i_m \delta^j_k - \delta^j_m \delta^i_k) e_k.$$

In the expression above, we use the Einstein summation convention, which is that we use both upper and lower indices, and we implicitly sum over any index which occurs both as an upper and a lower index. This convention allows us to write expressions without a lot of summation notation, since summation is implicit.

Any antisymmetric bilinear map $\psi : V \times V \to V$ can be expressed in the form

$$\psi = \psi^{ij}_k e_k,$$

where we sum over $i < j$. In particular, any Lie algebra structure can be expressed in this compact form.

Theorem 1.11. Suppose that $[\cdot, \cdot]$ is an antisymmetric bilinear map $V \times V \to V$, which satisfies the Jacobi identity on a basis. Then $[\cdot, \cdot]$ is a Lie algebra structure on $V$.

Proof. Let $V = \langle e_1, \cdots \rangle$ and suppose that

$$[e_i, [e_j, e_k]] = [[e_i, e_j], e_k] + [e_j, [e_i, e_k]]$$

for all $i, j, k$.

Let $x = a^i e_i$, $y = b^j e_j$, and $z = c^k e_k$. Using the bilinearity of the bracket, we obtain

$$[x, [y, z]] = [a^i e_i, [b^j e_j, c^k e_k]] = a^i b^j c^k [e_i, [e_j, e_k]]$$

$$= a^i b^j c^k ([e_i, e_j], e_k) + a^i b^j c^k [e_j, [e_i, e_k]]$$

$$= [[x, y], z] + [y, [x, z]].$$

This means that if $V$ is an $N$ dimensional vector space, then the Jacobi identity can be verified by testing $N^3$ equations, and since each of these equations has 3 terms, this means that we need to do $3N^3$ computations. In general, this is a very time consuming approach, so this is not a particularly powerful method of verifying that the Jacobi identity holds.

Example 2. On $\mathbb{K}^2$, the structure $\psi = \psi^{12}_2$ gives a Lie algebra structure. To see this, note that

$$[e_1, e_1] = 0, \quad [e_1, e_2] = e_2, \quad [e_2, e_1] = -e_2, \quad [e_2, e_2] = 0.$$

Exercise 8. Show that the example above satisfies the Jacobi identity. Note, you will have to compute $24 = 3(2^3)$ different combinations of the basis elements to show the Jacobi identity.

Example 3. There are some very important examples of matrix Lie algebras. They are subalgebras of the Lie algebra $\mathfrak{gl}(n, \mathbb{K})$, which is the Lie algebra structure on the associative algebra structure on the $n \times n$ matrices given by matrix multiplication. Some of these examples are given in the following list.
The special linear algebra $\mathfrak{sl}(n, K)$, given by
$$\mathfrak{sl}(n, K) = \{ A \in \mathfrak{gl}(n, K) | \text{Tr}(A) = 0 \}.$$

The algebra of upper triangular matrices $\mathfrak{t}(n, K)$, given by
$$\mathfrak{t}(n, K) = \{ (a_{ij}) \in \mathfrak{gl}(n, K) | a_{ij} = 0 \text{ if } i > j \}.$$

The algebra of strictly upper triangular matrices $\mathfrak{n}$, given by
$$\mathfrak{n} = \{ (a_{ij}) \in \mathfrak{gl}(n, K) | a_{ij} = 0 \text{ if } i \geq j \}.$$

The algebra of diagonal matrices $\mathfrak{d}(n, K) = \{ (a_{ij}) \in \mathfrak{gl}(n, K) | a_{ij} = 0 \text{ unless } i = j \}.$$

The special orthogonal algebra $\mathfrak{so}(n, \mathbb{R})$, given by
$$\mathfrak{so}(n, \mathbb{R}) = \{ A \in \mathfrak{gl}(n, \mathbb{R}) | A^T = -A \}.$$

The special unitary algebra $\mathfrak{su}(n) = \{ A \in \mathfrak{gl}(n, \mathbb{C}) | A^\ast = -A \}$, where $A^\ast$ is the conjugate transpose; i.e., $A^\ast = \bar{A}^T$.

**Exercise 9.** Prove that the above spaces are Lie algebras.

**Note:** It is customary to use *fraktur* notation for Lie algebras. For example, $\mathfrak{g}$ would stand for a vector space equipped with a Lie algebra structure.

**Definition 1.12.** A subspace $I$ of a Lie algebra $\mathfrak{g}$ is called an ideal if
$$[u, v] \in I \text{ if } u \in I \text{ and } v \in \mathfrak{g}.$$

**Exercise 10.** Let
$$T_k(n, K) = \{ A \in \mathfrak{gl}(n, K) | a_{ij} = 0 \text{ if } i + k < j \}.$$

Show that if $A \in T_k(n, K)$ and $B \in T_l(n, K)$, then $AB \in T_{k+l}(n, K)$. Moreover, show $[A, B] \in T_{k+l+1}(n, K)$. From this conclude that $T_l(n, K)$ is an ideal in $T_{l+1}(n, K)$ if $\ell \geq k$.

**Definition 1.13.** The lower central series $\mathfrak{g}^n$ of a Lie algebra $\mathfrak{g}$ is given by the recursive definition
- $\mathfrak{g}^0 = \mathfrak{g}$.
- $\mathfrak{g}^{n+1} = [\mathfrak{g}, \mathfrak{g}^n]$.

**Exercise 11.** Show that the descending central series is a sequence of ideals satisfying $\mathfrak{g}^0 \supseteq \mathfrak{g}^1 \supseteq \cdots$.

**Definition 1.14.** Suppose that $\mathfrak{g}$ is a Lie algebra such that $\mathfrak{g}^n = 0$ for some $n$. Then we say that $\mathfrak{g}$ is called a *nilpotent* Lie algebra.

**Definition 1.15.** The derived series $\mathfrak{g}^{(n)}$ of a Lie algebra $\mathfrak{g}$ is given recursively by
- $\mathfrak{g}^{(0)} = \mathfrak{g}$.
- $\mathfrak{g}^{(n+1)} = [\mathfrak{g}^{(n)}, \mathfrak{g}^{(n)}]$.

**Exercise 12.** Show that the derived series is a sequence of ideals satisfying $\mathfrak{g}^{(0)} \supseteq \mathfrak{g}^{(1)} \supseteq \cdots$.

**Definition 1.16.** Suppose that $\mathfrak{g}$ is a Lie algebra such that $\mathfrak{g}^{(n)} = 0$ for some $n$. Then $\mathfrak{g}$ is called a *solvable Lie algebra*. 
**Exercise 13.** Show that \( g^{(n)} \subseteq g^n \) for all \( n \). Conclude that any nilpotent Lie algebra is solvable.

**Exercise 14.** Consider the bracket defined on \( g = \langle x, y \rangle \) by the rule \([x, y] = y\). Show that this gives \( g \) the structure of a Lie algebra, and that this Lie algebra structure is solvable but not nilpotent.

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### 2. The Universal Enveloping Algebra

It is often advantageous to work with associative algebras rather than Lie algebras. In some cases, a Lie algebra arises from an associative algebra \( A \) by the bracket \([a, b] = ab - ba\), which equips \( A \) with the structure of a Lie algebra. However, most Lie algebras do not arise in this manner, so another construction is useful, which gives an associative algebra structure \( U(g) \), called the *Universal Enveloping Algebra* of \( g \). In order to construct this enveloping algebra we first need to define the tensor algebra of a vector space.

If \( V = \langle e_\lambda : \lambda \in \Lambda \rangle \), then a multi-index \( I \), of length \( n = |I| \) is an \( n \)-tuple \( I = (i_1, \ldots, i_n) \) where \( i_k \in \Lambda \) for all \( k \). We include the case \( n = 0 \), which is the 0-tuple \( () \). If \( I = (i_1, \ldots, i_n) \) and \( J = (j_1, \ldots, j_n) \) are two multi-indices, then \( IJ = (i_1, \ldots, i_m, j_i, \ldots, j_n) \) is their *concatenation*, which is a multi-index of length \( m + n \). Note that \((()I = I = I())\), for any multi-index \( I \).

**Definition 2.1.** The \( n \)-th tensor power \( T^n(V) \), also denoted by \( V^\otimes n \) (or just \( V^n \)) of a vector space \( V \) as above is given by \( T^n(V) = \langle e_I : |I| = n \rangle \). The *tensor algebra* \( T(V) \) of \( V \) is given by

\[
T(V) = \bigoplus_{n=0}^{\infty} T^n(V).
\]

There is a product defined on \( T(V) \) given by \( e_I e_J = e_{IJ} \) on basis elements, which is extended as a product on \( T(V) \) in the usual manner.

Note that if \( V \) is a finite dimensional vector space of dimension \( M \), then \( \dim(T^n(V)) = M^n \). This means that \( \dim(T(V)) = \infty \) unless \( V = 0 \).

**Exercise 15.** Show that \( T(V) \) is an *associative* algebra.

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