Hidden trees in the forest: On lattice points and prime labelings of graphs

Aba Mbirika*

Postdoctoral Fellow at Bowdoin College

September 2012

"The science of pure mathematics may claim to be the most original creation of the human spirit." –Alfred North Whitehead (1861-1947)

Abstract

Consider the integer lattice $\mathbb{Z}^2$ in the plane. It is well known that if a lattice point is selected at random, then the probability that it is visible from the origin is $\frac{6}{\pi^2}$. Can we find arbitrarily large squares of integer lattice points in which every point in this square is not visible from the origin? The answer is yes. The solution presented here uses systems of linear congruences solved by the Chinese Remainder Theorem. We then connect the concept of lattice point invisibility to a branch of graph theory involving prime labelings of the vertices of ladder graphs $P_n \times P_2$. Lastly, we conclude with some suggestions for undergraduate research projects related to these topics.

*The author thanks AIM-NSF for their REUF4 workshop at ICERM in June 2012. In Section 5, the author acknowledges the people there whose conversation and collaboration with him made this paper possible, and the five Bowdoin Science Experience students who engaged in a preliminary version of this paper.

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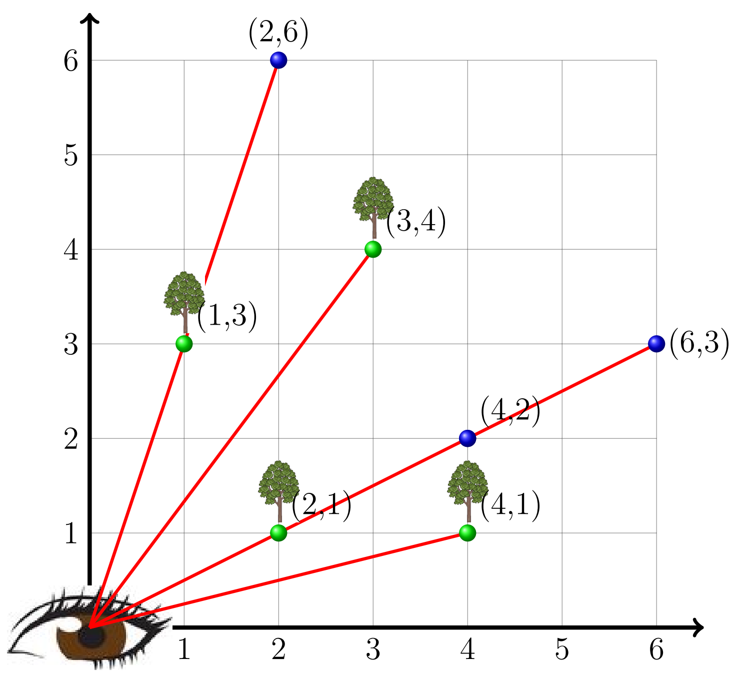
1 Visible lattice points from the origin?

Imagine the plane $\mathbb{R}^2$ as a forest in which each non-origin lattice point in $\mathbb{Z}^2$ is a tree. Imagining for a moment that each tree is infinitely thin, we say that a tree is hidden from your view at the origin if some other tree lies in your line of sight.



Consider the four lines of sight denoted by the red line segments emanating from the origin in the figure above. In these lines of sight only four trees are visible. These trees are located at the green bullet points at $(1,3)$, $(3,4)$, $(2,1)$, and $(4,1)$. Obscured by these trees are three other trees labeled by blue bullet points at $(2, 6)$, $(4, 2)$, and $(6, 3)$. The trees located at these blue points are not visible from the origin. The tree at $(2, 6)$ is obscured by the tree at $(1, 3)$, while the tree at $(6, 3)$ is obscured by the tree at $(4, 2)$ which in turn is obscured by the tree at $(2, 1)$. How are these last two integer coordinates related to the point $(6, 3)$?

The question of the visibility (or invisibility) of a lattice point from the origin can be recast in a number theoretic setting. The point $(2, 6)$ is invisible because the point $(1, 3)$ obscures it. And $(6, 3)$ is invisible since either $(4, 2)$ or $(2, 1)$ obscures it. It turns out that a point (x, y) is invisible from the origin if there is any point (x_0, y_0) such that $(x, y) = (cx_0, cy_0)$ for some $c \in \mathbb{N}_{>1}$. That is, if c divides both x and y, then (x, y) is not visible from the origin. Hence the only visible points are the points (x, y) such that $\gcd(x, y) = 1$.

A natural question to ask is what fraction of integer lattice points are visible from the origin? Let $T(n)$ equal the number of integer lattice points in an $n \times n$ square centered at the origin, and let $V(n)$ equal the number of these points visible from the origin. Then it suffices to compute the limit of $\frac{V(n)}{T(n)}$ as n approaches infinity. It turns out that this limit is $\frac{6}{\pi^2}$. One approach to a solution is to use the number theoretic characterization of visible lattice points given above. Proofs of this famous result are well known. Many involve the Möbius inversion formula and Euler-totient function (see Definition 3.12). In Theorem 1.3, we provide an alternative proof.

Remark 1.1 (Historical background to the problem)**.** The historical record of the original authorship of this result is inaccurately described on a number of occasions in the literature. Originally, the question on the probability of two random integers being coprime was raised in 1881 by Cesàro [2]. Two years later, he and Sylvester independently proved the result [3] and [8], respectively. Earlier in 1849, Dirichlet proved a slightly weaker form of the result [5]. The generalization to k coprime integers with $k > 2$ was presented again by Cesàro in 1884 [4]. This result was apparently proven independently in 1900 by Lehmer [7].

Remark 1.2. Since there is no uniform distribution on the natural numbers, it is somewhat imprecise to speak about the probability that two integers chosen at random are relatively prime. However, if we consider the uniform distribution on the set $\{1, 2, \ldots, n\}$ and then take the limit as n approaches infinity, then it is within this context in which we make any probability statements in the theorem below.

Theorem 1.3. *The fraction of pairs* (x, y) *in the integer lattice* $\mathbb{Z}^2$ *such that* $\gcd(x, y) = 1$ *is* $\dfrac{6}{\pi^2}$.

Proof. Let x and y be randomly selected integers. The probability that x is divisible by the prime p is $\frac{1}{p}$. Similarly y is divisible by p with probability $\frac{1}{p}$. By mutual independence, the probability that both x and y are divisible by p is $\frac{1}{p^2}$. Hence, the probability that both integers are not divisible by p is $1 - \frac{1}{p^2}$. For distinct primes, these divisibility events are mutually independent, thus the probability that no prime divides both x and y is the following product over the primes: $\displaystyle\prod_p \left(1 - \frac{1}{p^2}\right)$. To calculate this infinite product, it is helpful to consider the Riemann zeta function

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

for $s > 1$. A wonderful result of Euler connects this infinite sum with an infinite product of infinite sums over the primes. The essence of Euler's proof is its use of the fundamental theorem of arithmetic to observe that the sum can be written as the following infinite product

$$\sum_{n \geq 1} \frac{1}{n^s} = \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \cdots \right). \tag{1}$$

By multiplying out the product on the right side, each term $\frac{1}{n^s}$ on the left side appears exactly once, as a product of the appropriate powers of the primes in n. Since each product on the right hand side is a geometric series of the form $\dfrac{1}{1 - \frac{1}{p^s}}$, Equation (1) becomes

$$\sum_{n \geq 1} \frac{1}{n^s} = \prod_p \frac{1}{1 - \frac{1}{p^s}}.$$

Letting $s = 2$ and taking reciprocals, we get

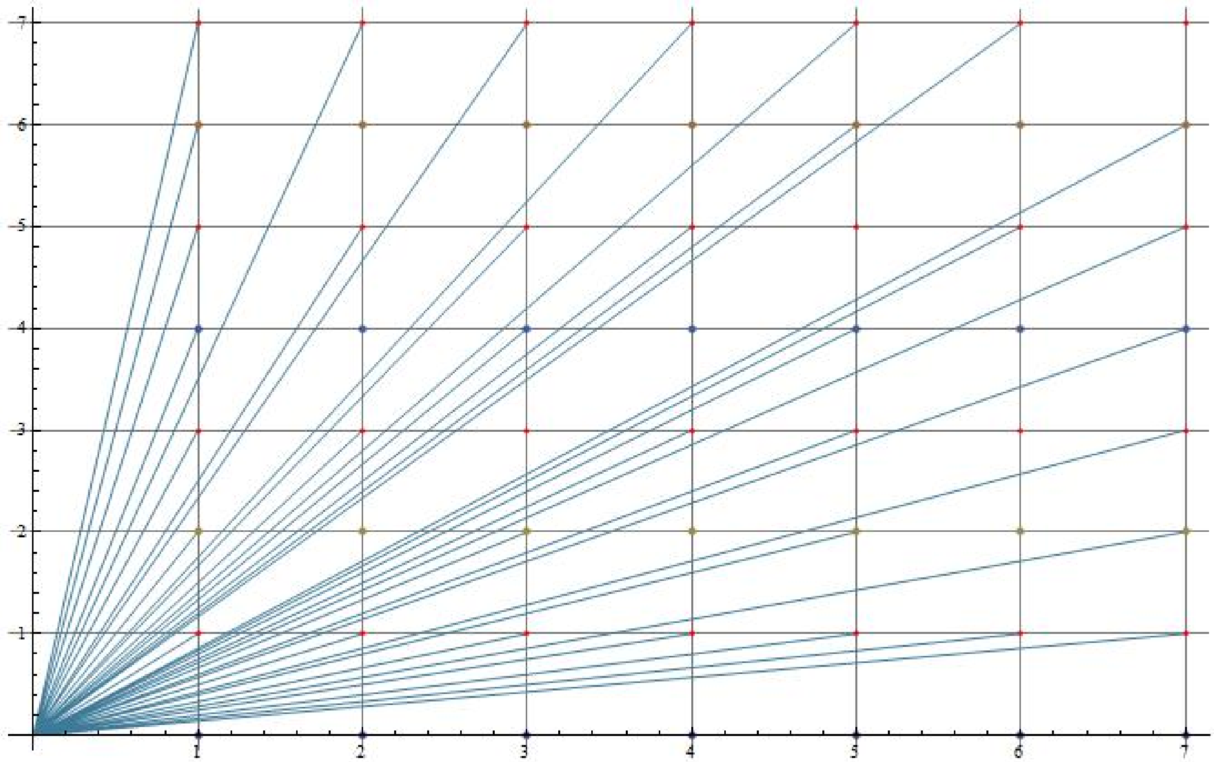
$$\frac{1}{\zeta(2)} = \prod_p \left(1 - \frac{1}{p^2}\right),$$

where the right side is the probability value we seek, and the left side is the reciprocal of the well-known evaluation of the Riemann zeta function at $s = 2$, namely $\zeta(2) = \frac{\pi^2}{6}$. Hence the fraction of lattice points (x, y) visible from the origin is $\frac{6}{\pi^2}$, as desired. □

Example 1.4. Consider the visible points in the interior of the 8×8 square below. The figure gives the first quadrant of the 15×15 square around the origin:

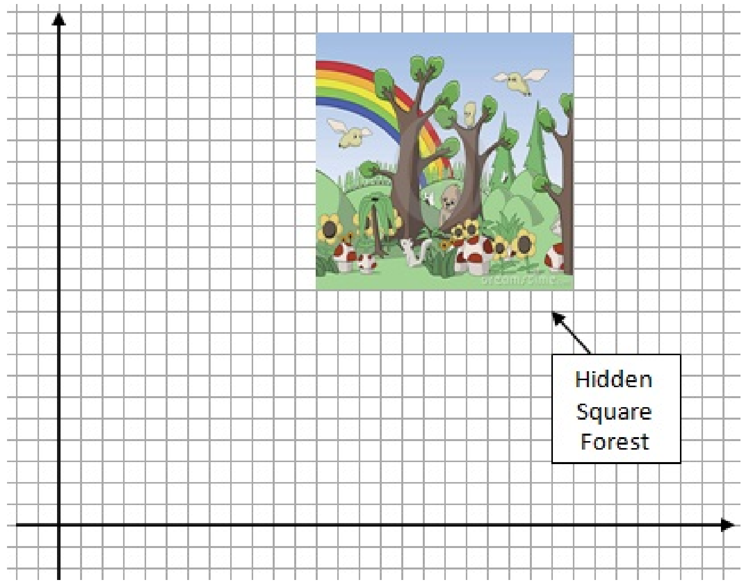
$$\Delta_7 = \{(x, y) \ : \ |x| \leq 7, \ |y| \leq 7\}.$$

In the figure, we can count that there are 35 visible points. By symmetry the square Δ_7 will have $35 \times 4 = 140$ visible points in addition to the four points $(\pm 1, 0)$ and $(0, \pm 1)$ on the x-axis and y-axis. Thus there are 144 visible points in Δ_7. Hence the fraction of visible points is $\frac{144}{225} = 0.64$, which is within .032 of the predicted fraction $\frac{1}{\zeta(2)}$ given in Theorem 1.3.



2 Arbitrarily large patches of invisible lattice points?

Again suppose that you are standing at the origin in the integer lattice of trees. It seems very likely that the closer one looks from the origin, then the more trees one can see. Consequently, one might imagine that the further away one looks, the more trees there are that are obscured from view. A natural question to ask is the following: Are there arbitrarily large square patches that are not visible from the origin? Can we discover *hidden forests* of arbitrarily large sizes? Where exactly are they? Can we extend this to higher dimensions? The answer to these questions is yes, and the solution involves a wonderful use of linear congruences and the Chinese Remainder Theorem. We solve this question in the two-dimensional setting and then invite the reader to extend it to higher dimensions (see Project 4.3).



Theorem 2.1. *For every $n \in \mathbb{N}$, there exists disjoint sets A_1 and A_2 each containing n consecutive natural numbers such that* $\gcd(a_1, a_2) > 1$ *whenever $a_i \in A_i$.*

Proof. Let $\{p_1, p_2, \ldots\}$ be the set of primes. Construct the matrix containing the first n^2 primes by filling row i with the primes $p_{(i-1)n+1}$ through $p_{(i-1)n+n}$ to yield the matrix:

$$\begin{pmatrix} p_1 & p_2 & p_3 & \cdots & \cdots & p_j & \cdots & \cdots & p_n \\ p_{n+1} & p_{n+2} & p_{n+3} & \cdots & \cdots & p_{n+j} & \cdots & \cdots & p_{2n} \\ p_{2n+1} & p_{2n+2} & p_{2n+3} & \cdots & \cdots & p_{2n+j} & \cdots & \cdots & p_{3n} \\ \vdots & \vdots & \vdots & & & \vdots & & & \vdots \\ \vdots & \vdots & \vdots & & & \vdots & & & \vdots \\ p_{(i-1)n+1} & p_{(i-1)n+2} & p_{(i-1)n+3} & \cdots & \cdots & p_{(i-1)n+j} & \cdots & \cdots & p_{(i-1)n+n} \\ \vdots & \vdots & \vdots & & & \vdots & & & \vdots \\ \vdots & \vdots & \vdots & & & \vdots & & & \vdots \\ p_{(n-1)n+1} & p_{(n-1)n+2} & p_{(n-1)n+3} & \cdots & \cdots & p_{(n-1)n+j} & \cdots & \cdots & p_{n^2} \end{pmatrix}.$$

Let R_i and C_j be the product of the entries in row i and column j, respectively, so we have

$$R_i = (p_{(i-1)n+1}) \cdot (p_{(i-1)n+2}) \cdot (p_{(i-1)n+3}) \cdots (p_{(i-1)n+n}), \quad \text{and}$$
$$C_j = (p_j) \cdot (p_{n+j}) \cdot (p_{2n+j}) \cdot (p_{3n+j}) \cdots (p_{(n-1)n+j}).$$

Since they share no primes in common, the row products $R_1, R_2, \ldots, R_n$ are pairwise relatively prime. Similarly, the column products $C_1, C_2, \ldots, C_n$ are pairwise relatively prime. Consider the following pair of systems of linear congruences:

$$\begin{aligned} x+1 &\equiv 0 \pmod{R_1} \\ x+2 &\equiv 0 \pmod{R_2} \\ &\vdots \\ x+n &\equiv 0 \pmod{R_n} \end{aligned} \qquad\qquad \begin{aligned} y+1 &\equiv 0 \pmod{C_1} \\ y+2 &\equiv 0 \pmod{C_2} \\ &\vdots \\ y+n &\equiv 0 \pmod{C_n}. \end{aligned}$$

By the Chinese Remainder Theorem, there exists solutions x_0 and y_0 to the left and right systems, respectively, such that x_0 is unique modulo the integer $R_1 \cdot R_2 \cdots R_n$ and y_0 is unique modulo the integer $C_1 \cdot C_2 \cdots C_n$. Observe that these integers are the product of all entries in the matrix so we have $R_1 \cdot R_2 \cdots R_n = C_1 \cdot C_2 \cdots C_n = \prod_{i=1}^{n^2} p_i$, which we denote M. Let $A_1 = \{x_0+1, x_0+2, \ldots, x_0+n\}$ and $A_2 = \{y_0+1, y_0+2, \ldots, y_0+n\}$. We claim that none of the elements in A_1 are pairwise relatively prime to any of the elements in A_2. For an arbitrary $x_0+i \in A_1$ and $y_0+j \in A_2$, these elements by construction are multiples of R_i and C_j, respectively, and hence their greatest common divisor equals the common prime that lies in the intersection of row i and column j in the matrix—that is, $\gcd(x_0+i, y_0+j) = p_{(i-1)n+j} > 1$ as desired.

If the sets A_1 and A_2 are disjoint, then we are done. Otherwise, by simply adding M to each of the n elements of A_2, we get the following new set that is clearly disjoint from A_1:

$$A_2 = \{y_0+1+M, y_0+2+M, \ldots, y_0+n+M\}.$$

It is also clear that $\gcd(x_0+i, y_0+j+M)$ still equals $p_{(i-1)n+j}$, and the two-dimensional case is done. □

Example 2.2. Let us consider the simplest case of a 2×2 hidden forest. Using Theorem 2.1, if we choose n to be 2, then the prime matrix is

$$\begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix}.$$

The row products are $R_1 = 6$ and $R_2 = 35$, while the column products are $C_1 = 10$ and $C_2 = 21$. Hence the corresponding linear congruences we need to solve are

$$\begin{aligned} x+1 &\equiv 0 \pmod{6} \\ x+2 &\equiv 0 \pmod{35} \end{aligned} \qquad\qquad \begin{aligned} y+1 &\equiv 0 \pmod{10} \\ y+2 &\equiv 0 \pmod{21}. \end{aligned}$$

By the Chinese Remainder Theorem, the left and right systems have the unique solutions

$$x_0 = 173 \pmod{210} \text{ and}$$
$$y_0 = 19 \pmod{210},$$

respectively. Let $A_1 = \{174, 175\}$ and $A_2 = \{20, 21\}$. Then $A_1 \cap A_2 = \emptyset$ and $\gcd(a_1, a_2) > 1$ for each $a_i \in A_i$. Thus, there is a hidden forest of four trees at $(174, 20)$, $(174, 21)$, $(175, 20)$, and $(175, 21)$. In the figure below we draw the hidden forest, and to its right we draw the four nodes again but highlight the greatest common divisors of the coordinates of each node.

Observe that the gcd-figure on the right resembles the prime matrix up to a permutation of the entries. In fact we see that our original matrix is

$$\begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix} = \begin{pmatrix} \gcd(x_0+1, y_0+2) & \gcd(x_0+2, y_0+2) \\ \gcd(x_0+1, y_0+1) & \gcd(x_0+2, y_0+1) \end{pmatrix}. \tag{2}$$

Question 2.3. *For such a small square of trees, the hidden forest in the previous example seems very far from the origin! Can you find a closer hidden forest? In Figure 3.1, we give the* 13×13 *first quadrant lattice* $\{(x, y) \,:\, 0 \le x, y \le 13\}$ *and place a bullet at each invisible point. This figure confirms that there are no* 2×2 *hidden forests in at least this subset of the integer lattice.*

Example 2.4. Use Theorem 2.1 to calculate the location of the sixteen trees in a 4×4 hidden forest. Note: if done correctly, then x_0 equals 2,847,617,195,518,191,809 and y_0 equals 1,160,906,121,308,397,397. It seems that although the algorithm in the theorem yields a hidden forest, it may not yield a forest remotely "close" to the origin.

Question 2.5. *What happens if we switch around some of the 4 entries in the prime matrix in the first example? Or choose different primes?*

We now generalize Theorem 2.1 to arbitrary d-dimensional integer lattices and leave the proof of this as an exercise for the reader (see Project 4.3).

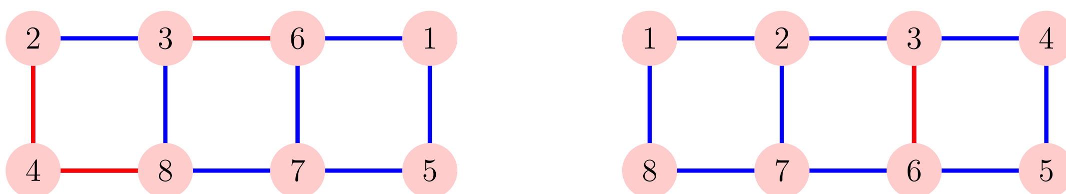
Theorem 2.6. *For every* $n, d \in \mathbb{N}$, *there exists disjoint sets* $A_1, A_2, \ldots, A_d$ *each containing* n *consecutive natural numbers such that* $\gcd(a_1, a_2, \ldots, a_d) > 1$ *whenever* $a_i \in A_i$.

3 A connection to prime labelings of graphs

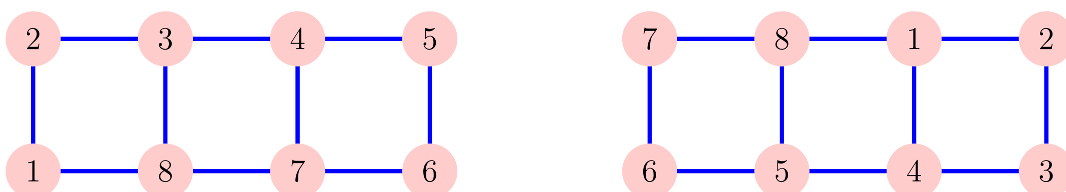
This section contains some unpublished results of the author and his collaborators [1].

Definition 3.1. A *prime labeling* of a graph G is a labeling of the vertices with the integers $1, 2, \ldots, v$ (where v is the number of vertices) such that any two adjacent vertices have labels that are relatively prime. We call a labeling of G a *coprime labeling* if the labeling set comes from the integers $1, 2, \ldots, m$ for some $m \geq v$. Denote $Pr(G)$ to be the smallest m that makes the labeling possible for G. If $Pr(G) = v$, then we say the graph G is prime.

Example 3.2. Consider the ladder graph $P_4 \times P_2$. The two vertex labelings given below are not prime labelings since the edges, which we denote in red, are connecting vertices whose labels are not relatively prime.



Below, however, are two non-isomorphic prime labelings of $P_4 \times P_2$.



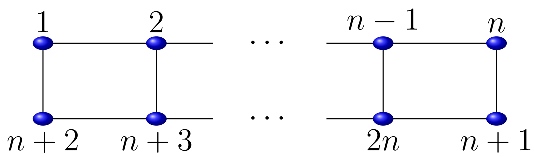
The consecutive numbering around the edges of the ladders gives the prime labelings in these last two examples a particularly nice presentation. Hence we give them a special name.

Definition 3.3. We say that a ladder graph $P_n \times P_2$ has a *consecutive cyclic prime labeling* if the numbers $1, 2, \ldots, 2n$ can be placed cyclically and in that order around the vertices of the ladder such that the ladder is prime.

Question 3.4. *It is clear that in any consecutive cyclic (not necessarily prime) labeling for* $P_n \times P_2$ *that the sum of the vertex labels for each pair of vertices associated to a vertical edge equals the same number modulo* $2n$. *For instance in the* $n = 4$ *case of Example 3.2, the sum in the top right one is* $9 \pmod 8$*, the bottom left one is* $11 \pmod 8$*, and the bottom right one is* $13 \pmod 8$. *These number modulo 8 are 1, 3, and 5, respectively. Is there a rhyme or reason to why sums 3 and 5 work, but sum 1 gives a bad labeling?*

Theorem 3.5 (Berliner, et al. [1]). *If $n+1$ is prime, then $P_n \times P_2$ has a prime labeling. Moreover, this prime labeling can be realized with top row labels from left to right,* $1, 2, \ldots, n$, *and bottom row labels from left to right,* $n+2, n+3, \ldots, 2n, n+1$.

Proof. Consider the graph $P_n \times P_2$ where $n+1$ is prime. We claim that the following vertex labeling gives a prime labeling:



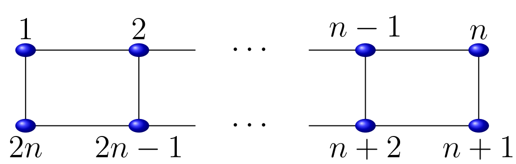
Since $\gcd(k, k+1) = 1$, it suffices to check only the vertex labels arising from the endpoints of the following n particular edges:

- the horizontal edge connecting vertex labels $2n$ and $n+1$, and
- the first $n-1$ vertical edges going from left to right.

Since $n+1$ is prime and $2n < 2(n+1)$, then $n+1$ cannot divide $2n$. Hence $\gcd(2n, n+1) = 1$ as desired. Observe that each of the $n-1$ vertical edges under consideration have vertex labels a and $(n+1)+a$ for $1 \leq a \leq n-1$. It follows that $\gcd((n+1)+a, a) = \gcd(n+1, a) = 1$. Thus the graph $P_n \times P_2$ is prime whenever $n+1$ is prime. □

Theorem 3.6 (Berliner, et al. [1]). *If $2n+1$ is prime, then $P_n \times P_2$ has a consecutive cyclic prime labeling which can be realized by assigning the label* 1 *to the top left vertex of the graph.*

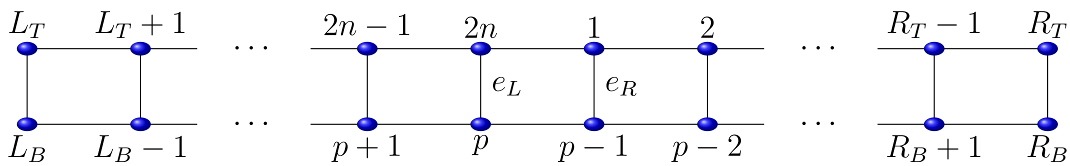
Proof. Consider the graph $P_n \times P_2$ where $2n+1$ is prime. We claim that the following vertex labeling gives a consecutive cyclic prime labeling:



Since $\gcd(k, k+1) = 1$, it suffices to check only the vertex labels arising from the endpoints of the first $n-1$ vertical edges going from left to right. Observe that each of the $n-1$ vertical edges under consideration have vertex labels a and $(2n+1)-a$ for $1 \leq a \leq n-1$. We conclude that $\gcd(a, (2n+1)-a) = \gcd(a, 2n+1) = 1$. Thus the graph $P_n \times P_2$ has a consecutive cyclic prime labeling whenever $2n+1$ is prime. □

Theorem 3.7 (Berliner, et al. [1]). *If $2n+p$ is prime such that p is a prime less than $2n+1$, then $P_n \times P_2$ has a consecutive cyclic prime labeling. Moreover, this labeling can be realized by assigning 1 to the vertex in the location $\frac{1}{2}(p-1)-1$ places from the top right vertex.*

Proof. Let $2n+p$ be a prime such that p is prime and less than $2n+1$. Consider the graph $P_n \times P_2$. We claim that the following vertex labeling gives a consecutive cyclic labeling:



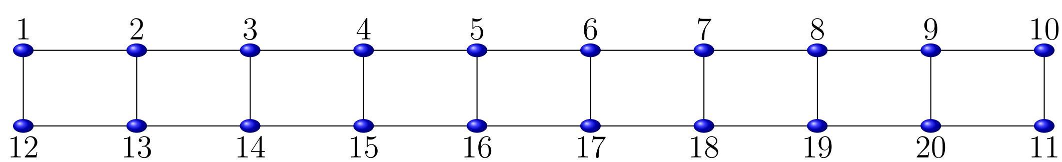
where $L_T = \frac{1}{2}(p+1)+n$, $L_B = \frac{1}{2}(p-1)+n$, $R_T = \frac{1}{2}(p-1)$, and $R_B = \frac{1}{2}(p-1)+1$. Since $\gcd(k, k+1) = 1$, it suffices to check only the vertex labels arising from the endpoints of the first $n-1$ vertical edges going from left to right. There are two cases to consider:

- the vertical edges right of (and including) edge e_R, and
- the vertical edges left of (and including) edge e_L.

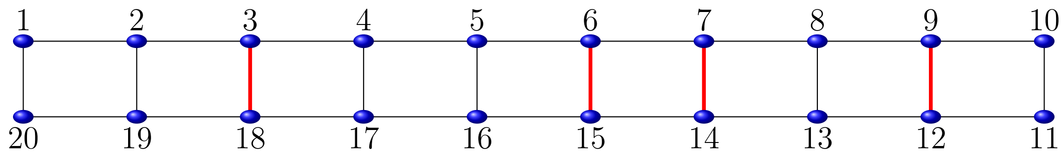
For the first case, observe that each of the edges under consideration have vertex labels a and $p-a$ for $1 \leq a < \frac{1}{2}(p-1)$. It follows that $\gcd(a, p-a) = \gcd(a, p) = 1$. For the second case, observe that each of the edges under consideration have vertex labels a and $(2n+p)-a$ for $p \leq a \leq \frac{1}{2}(p-1)+n$. We conclude that $\gcd(a, (2n+p)-a) = \gcd(a, 2n+p) = 1$. Thus the graph $P_n \times P_2$ has a consecutive cyclic prime labeling whenever $2n+p$ is prime and p is a prime less than $2n+1$. □

Question 3.8. *For any integer n, is there a prime $2n+p$ where either $p = 1$ or p is a prime and less than $2n+1$? If the answer to this is yes, then as a result of Theorem 3.6 and Theorem 3.7, we may conclude that not only is $Pr(P_n \times P_2) = 2n$ for all n but also that every ladder has a consecutive cyclic prime labeling which we can easily construct.*

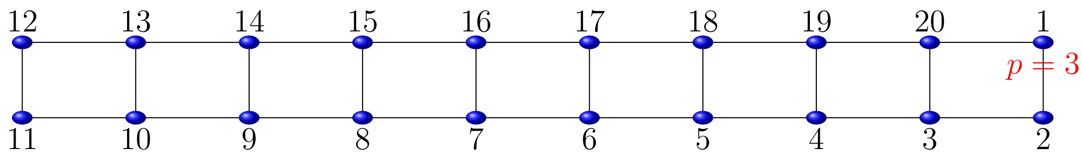
Example 3.9. Consider the graph $P_{10} \times P_2$. Here $n = 10$ and thus $n+1$ is prime so Theorem 3.5 holds and we have the following prime labeling:

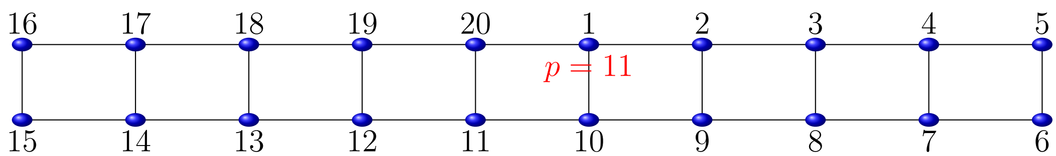


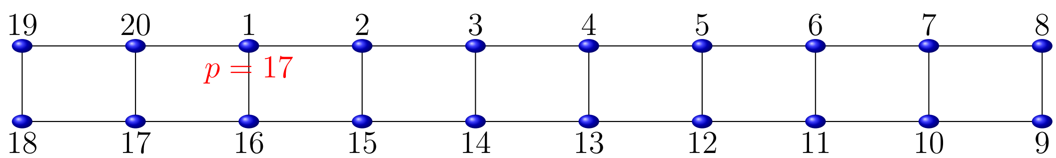
Theorem 3.6 does not apply since $2n + 1 = 21$ and hence is not prime. Further we see that assigning 1 to the top left vertex does not yield a consecutive cyclic prime labelings since the edges, which we denote in red, are connecting vertices whose labels are not relatively prime.



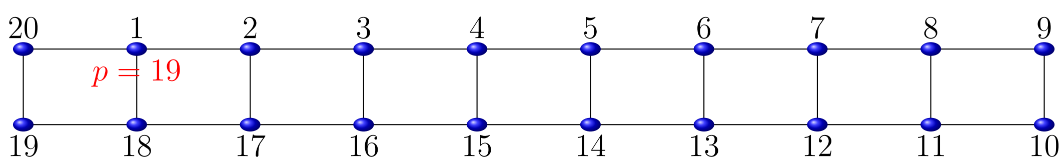
Lastly, out of all the primes p such that $p < 2n + 1$, only the primes 3, 11, and 17 yield values $2n + p$ which are prime—namely 23, 31, and 37, respectively. Theorem 3.7 holds and we have the following three consecutive cyclic prime labelings:







In each labeling we highlight the p-value that corresponds to exactly where we need to assign the value 1 (as determined by Theorem 3.7). For $p = 19$ we observe that $2n + p = 39$ is not prime, yet the following labeling shows that assigning 1 to the prescribed vertex gives a successful labeling nonetheless.



An exhaustive check shows that assigning 1 to any other vertex in the top row fails to yield a consecutive cyclic prime labeling.

Question 3.10. *The previous example proves that the converse of Theorem 3.7 does not hold. That is, there are primes p for which $2n + p$ is not prime, yet the prescribed labeling is successful. However this same example does not disprove the converse of Theorem 3.6. Does there exist an n such that $P_n \times P_2$ has a successful labeling with the value 1 assigned to the top left vertex, but $2n + 1$ is not prime?*

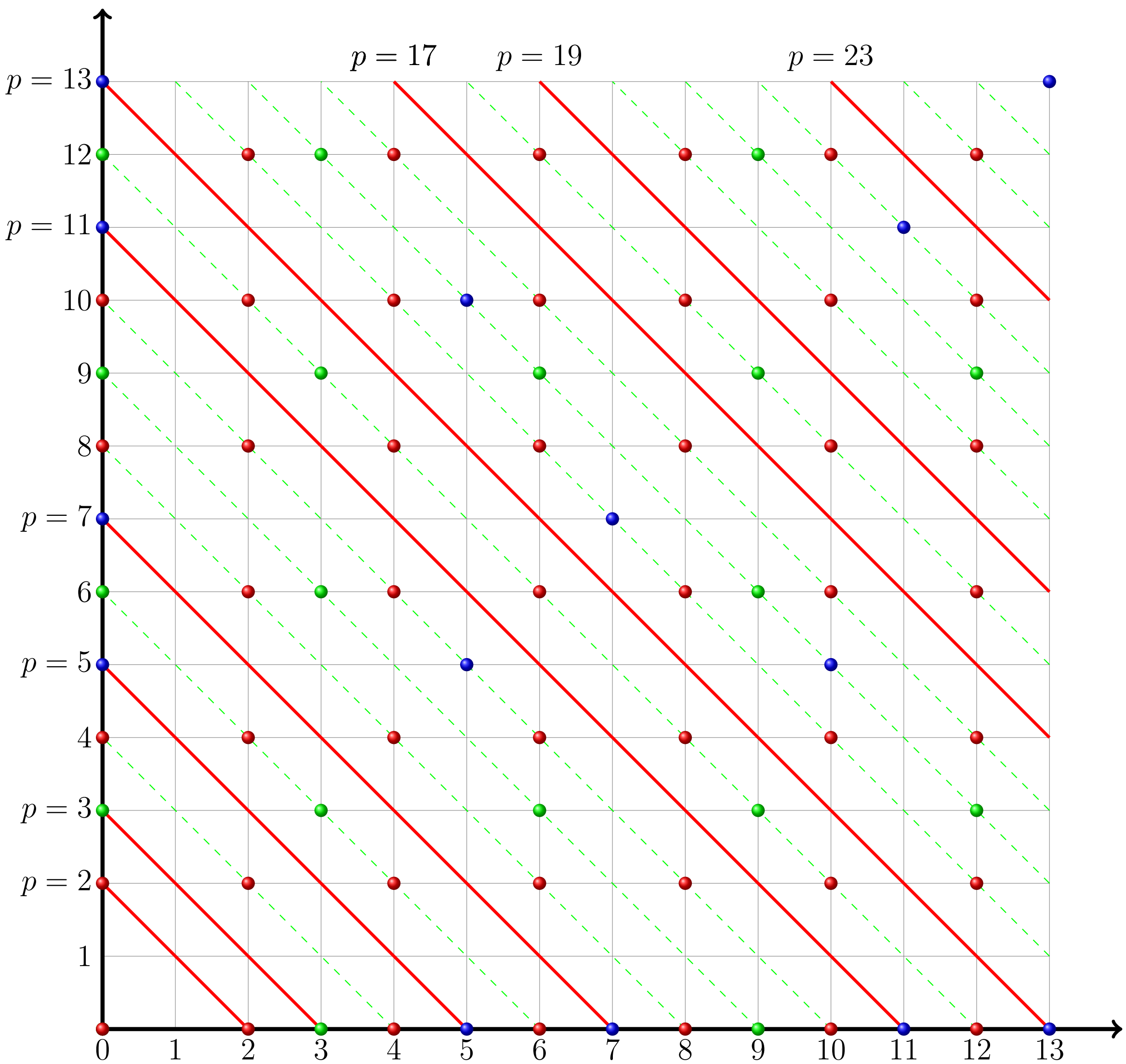


Figure 3.1: Points (shown as bullets) not visible from the origin

In Figure 3.1, we show a connection between certain visible points from the origin and lines (shown in red) of the form $y = -x + p$ where p is a prime. The main point of this section is to connect prime labelings of graphs to lines of slope -1 with prime y-intercept values. First we ask the reader to explore the following question.

Question 3.11. *Fix a prime p. Prove or disprove the following: All of the lattice points in which the line $y = -x + p$ passes through in the interior of the first quadrant are visible points.*

Definition 3.12 (Euler's totient function)**.** For $n \geq 1$, let $\phi(n)$ denote the number of positive integers not exceeding n that are relatively prime to n.

The totient function is beneficial to us as it counts the number of integers $k \leq n$ such that $\gcd(n, k) = 1$. Some useful properties of the totient function are the following:

- $\phi(n) = n - 1$ if and only if n is prime.
- If p is prime and $k > 0$, then $\phi(p^k) = p^k - p^{k-1} = p^k \left(1 - \frac{1}{p}\right)$.
- ϕ is a multiplicative function (i.e., $\phi(mn) = \phi(m)\phi(n)$ whenever $\gcd(m, n) = 1$).

The following theorem combines the last two properties to yield a practical method of computing the value of the totient function of any positive integer.

Theorem 3.13. *Let $n \geq 1$ be an integer with prime factorization $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$. Then*

$$\begin{aligned}\phi(n) &= (p_1 - p_1^{k_1 - 1})(p_2 - p_2^{k_2 - 1}) \cdots (p_r - p_r^{k_r - 1}) \\ &= n \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_r}\right).\end{aligned}$$

Proof. Exercise left for the reader. Hint: Induct on r. □

Question 3.14. *For $n \geq 2$ it appears that the number of visible points through which the line $y = -x + n$ passes in the interior of the first quadrant is exactly $\phi(n)$. Can you explain why? Observe that Question 3.11 is a special case of this question.*

Question 3.15. *Consider the line $y = -x + 9$ in Figure 3.1. Is there structure in the gaps between invisible points? These gaps seem to be related to the values which are relatively prime to 9. That is, these $\phi(9) = 6$ values are spaced like so*

$$\not{0} \quad 1 \quad 2 \quad \not{3} \quad 4 \quad 5 \quad \not{6} \quad 7 \quad 8 \quad \not{9},$$

where we strike through values k such that $\gcd(k, 9) > 1$. The values not striked through are exactly the x-values on the line $y = -x + 9$ for which the point (x, y) is a visible lattice point. Does this pattern hold for all lines $y = -x + n$?

4 Possible undergraduate research projects

The first three projects involve exploring concepts that are already known. For the remaining projects, many of the answers to these questions are not yet known. These are called open problems. It is your job to find answers or make some headway in these exercises. This is the heart of mathematical research. Perhaps in attempting to answer any of these open problems, you will come up with answers to different questions that arise along the way. Or instead of precise answers to the questions, you might come up with conjectures on different perspectives of the problems. Be creative and enjoy the exploration. Research in mathematics can be a lot of fun.

Project 4.1. Research the Basel problem. Find some resources that explain exactly what this problem is and sketch at least one proof. For instance, the following paper by Robin Chapman gives 14 different proofs of this one problem. See `http://empslocal.ex.ac.uk/people/staff/rjchapma/etc/zeta2.pdf` (Note: Most of these proofs require calculus and/or higher levels of math).

Project 4.2. In Theorem 2.1, we made extensive use of the Chinese Remainder Theorem (CRT). Find some resources that explain exactly what CRT is. The Wikipedia entry is a decent starting point. It poses the question: "What is the lowest number n that when divided by 3 leaves a remainder of 2, when divided by 5 leaves a remainder of 3, and when divided by 7 leaves a remainder of 2?" Try first to answer this question without first looking at CRT or any outside resources. Perhaps you might stumble upon the essential working ideas of CRT by yourself.

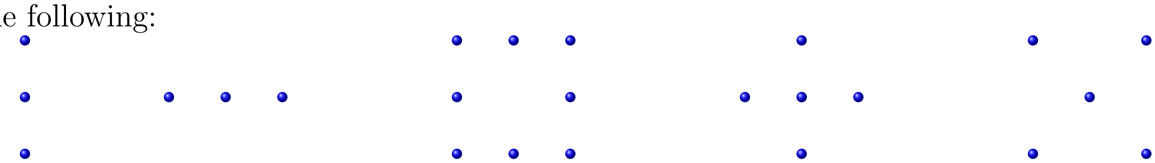
Project 4.3. Theorem 2.1 is just the base case for higher dimensional analogs of invisible square forests. For dimension $d = 3$, are there arbitrarily large cubes of hidden forests in the integer lattice $\mathbb{Z}^3$? The answer is yes! How about for any dimension d? That is confirmed in Theorem 2.6, but we leave the proof to you. Colleague Stephan Garcia of Pomona College suggests the following starting point:

- Use a d-dimensional array of the first n^d prime numbers. You would get d systems of congruences, where the moduli would be "slices" of the array (e.g., $(d-1)$-dimensional "subspaces"). Each of the systems of congruences is solvable by CRT since the products of the slices are relatively prime. For getting the gcd of a d-tuple to be greater than 1, think about planes in $\mathbb{R}^3$ parallel to the x, y, z axes. Intersecting three of them (each parallel to a distinct axis) yields a single point. The same idea gives you a single prime which divides every element of the d-tuple.

Try to use Garcia's advice as a guideline and carefully think through each step of this generalization of the 2×2 case solved in Theorem 2.1. The generalization to Theorem 2.6 truly is a wonderful result that you can proudly say you have solved.

Project 4.4. Find the 3×3 hidden forest in the first quadrant of the integer lattice that is yielded by the method given in the proof of Theorem 2.1. A 2×2 example is given in Example 2.2. Also find which entries of the prime matrix correspond to which coordinates' greatest common divisors as in Equation (2) of the 2×2 example. Can you use this information to predict what the prime matrix would correspond to for the $n = 4$ case and higher?

Project 4.5. Can you find a 3×3 forest closer than the one yielded by Project 4.4? As an easier first exercise, try finding closer subsets of a 3×3 hidden forest. Try subsets such as the following:



As an alternate project, perhaps focus only the first two types of subsets above. Consider these for various values of n starting with $n = 2$. Come up with some ideas on how to find a hidden line of trees for successively larger values of n. Can you extend these ideas to an algorithm for finding arbitrary length n lines of hidden trees?

Project 4.6. There are six questions in Section 3 on the prime labelings of ladder graphs. These are open questions and hence there may or may not be straight-forward answers as such. Explore these questions and make some observations as a scientist would do in a lab. Question 3.8 might be the hardest to tackle but could lead to other interesting questions. Question 3.4 seems very intriguing, and it might be best to start with many small examples. Question 3.10 seems doable, but again try many small examples first. Questions 3.11 and 3.14 are very interrelated, so doing one leads you to the other problem. Lastly, Question 3.15 might be the easiest to begin. It might also help with the previous two questions before it (should you also choose to attempt those).

Project 4.7. In Section 3, we considered the prime labelings of ladder graphs in particular. Consider doing this analysis on different families of graphs; for example, cycles, wheels, stars, paths, caterpillars, friendship graphs, fullerene graphs, generalized Petersen graphs (also known as prism graphs), etc. A gallery of named graphs with helpful illustrations is given at http://en.wikipedia.org/wiki/Gallery_of_named_graphs. A very helpful resource by Joseph Gallian gives an updated survey of this very active area of research [6]. The latest update was published in 2011 and is freely available at http://www.combinatorics.org/ojs/index.php/eljc/article/view/ds6/pdf.

5 Acknowledgments

The author Aba Mbirika thanks Stephan Garcia of Pomona College who introduced him to the content of Section 2 at the AIM-NSF research workshop, REUF4, at ICERM in June 2012. The author especially thanks his research collaborators there: Adam Berliner of St. Olaf College, Nate Dean of Texas State University, Jonelle Hook of Mount St. Mary's University, Alison Marr of Southwestern University, and Cayla McBee of Providence College, with whom he worked on the (as of yet) unpublished results on prime labelings of ladder graphs contained in Section 3. Lastly, thank you to Bowdoin College and my five students (Wilfed Ahoua, Sergio Gomez, Tess Lameyer, Brian Moore, and lab assistant Rachel Pollinger) who were the first to engage in a preliminary version of this paper and the research therein at the Summer 2012 Bowdoin Science Experience's mathematics section.

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