Definitions of a permutation in $S_n$ and a generalized permutation in $G(r, 1, n)$:

Define $[n]$ to be the set $\{1, 2, ..., n\}$. A permutation is any arrangement of the elements of $[n]$. Clearly, there are $n! = 1 \times 2 \times 3 \times \cdots \times n$ possible permutations. The symbol $n!$ is pronounced “n-factorial” and is used to denote the latter product. We call the set of all possible permutations of $[n]$ the symmetric group and denote it $S_n$.

An element $\sigma$ in $S_n$ can be viewed in one-line notation as $\sigma_1 \sigma_2 \cdots \sigma_n$ where $\sigma_i := \sigma(i)$ is the value where $\sigma$ sends the value $i$. For example, in $S_3$ the element 1 3 2 is the permutation that fixes 1 (that is, $\sigma(1) = 1$) and switches the values 2 and 3 (that is, $\sigma(2) = 3$ and $\sigma(3) = 2$).

The symmetric group can be generalized to a larger set of objects in a group that we call the complex reflection groups, $G(r, p, n)$. When $p = 1$, the elements of this group are called generalized (or colored) permutations. An element $w$ in $G(r, 1, n)$ can be viewed in window notation as $[\zeta^{a_1} \sigma_1, \zeta^{a_2} \sigma_2, ..., \zeta^{a_n} \sigma_n]$ where $\zeta$ is a primitive root of unity (that is, $\zeta$ is a complex number such that $\zeta^r = 1$ and $\zeta^k \neq 1$ for any positive integer $k < r$), and $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n$ is an element in the symmetric group $S_n$. The exponents $a_1, a_2, ..., a_n$ are all integers between 0 and $r - 1$. Hence, it is clear that there are $r^n \times n!$ possible permutations in $G(r, 1, n)$. If we set $r = 1$, then we recover the symmetric group $S_n$ from the complex reflection groups $G(r, 1, n)$. If we set $r = 2$, we get the hyperoctahedral group which is the well-known symmetry group of the $n$-cube in $n$-dimensional space.

Motivation for the research project / Connection to previous ORSP project:

During my postdoc at Bowdoin College from 2010 to 2013, I conducted research on $G(r, p, n)$ [1,5]. In January of 2016, I published a paper that is the result of the work done there [2]. In the Summer of 2016, I received an SREU grant to explore one specific permutation statistic on the group $G(r, 1, n)$—specifically, we are exploring
the topic of the longest increasing subsequences in these generalized permutations. This preliminary summer research has led to a host of open problems in this area. The length of a longest increasing subsequence is an example of one type of permutation statistic. More precisely, a **permutation statistic** is a function that inputs a permutation and outputs a non-negative integer. Two well-studied statistics on $S_n$ are the descents and the inversions. Let $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ be a permutation in $S_n$. A **descent** in $\sigma$ is a position $i$ such that $\sigma_i > \sigma_{i+1}$ and an **inversion** in $\sigma$ is any pair $(\sigma_i, \sigma_j)$ such that $i < j$ and $\sigma_i > \sigma_j$. These two statistics are intimately related to two other statistics we call excedance and major index. We say that a position $i$ is an **excedance** in $\sigma$ if $\sigma_i > i$. The **major index** of $\sigma$ is the sum of the descent positions, and we denote this number $\text{maj}(\sigma)$. We use the notations $\text{Des}(\sigma)$, $\text{Inv}(\sigma)$, and $\text{Exc}(\sigma)$ to denote the three sets of descents, inversions and excedances, and small caps $\text{des}(\sigma)$, $\text{inv}(\sigma)$, and $\text{exc}(\sigma)$ to denote the sizes of each corresponding set. For example, let $\sigma = 3 \ 5 \ 2 \ 4 \ 6 \ 1$ be an element in the group $S_6$. Then $\text{Des}(\sigma) = \{2, 5\}$, $\text{Inv}(\sigma) = \{(3, 2), (3, 1), (5, 2), (5, 1), (4, 1), (2, 1), (6, 1)\}$, $\text{Exc}(\sigma) = \{1, 2, 5\}$, and $\text{maj}(\sigma) = 7$. For all nonnegative integers $n$ and $k$, we have the following remarkable facts:

- **Fact 1**: The number of permutations in $S_n$ with $k$ inversions equals the number of permutations with major index $k$.
- **Fact 2**: The number of permutations in $S_n$ with $k$ descents equals the number of permutations with $k$ excedances.

The statistical pairs $(\text{inv}, \text{maj})$ and $(\text{des}, \text{exc})$ are each said to be **equidistributed statistics**. Moreover, any statistic equidistributed with those in Fact 1 is called **Mahonian**, while any statistic equidistributed with those in Fact 2 is called **Eulerian**. The subject of permutation statistics has been an active and important area of enumerative combinatorics over the last four decades. But only recently have people looked at application of these permutation statistics to generalized permutations in the group $G(r, 1, n)$. That is what my research students are currently doing this summer as I write this ORSP Research Proposal.

**Project description**

In the past 10 years, much work has been done on generalizing Eulerian and Mahonian statistical analogues of the $S_n$-results to the new setting of $G(r, 1, n)$. However in my recent paper [2], we compute the sign statistic for
elements in $G(r, 1, n)$ in a new way—namely, we view each $w$ in $G(r, 1, n)$ as a pair of $r$-multitableaux (see my expository paper [3] for a precise definition). Our goal in this 2016-2017 research project is to explore if it is possible to extract other well-known permutation statistics not from the permutations themselves (which is recently well-studied) but from the permutation presentations as a pairs of $r$-multitableux. Some of these results from the symmetric group setting that might generalize are as follows:

**QUESTION 1:** In the symmetric group setting, if $\sigma$ in $S_n$ is an involution (i.e., $\sigma$ is its own inverse), then the entries in the Young tableaux under the Schensted correspondence (see [3] and [4]) are identical. This phenomenon does not occur in the $G(r, 1, n)$ setting. Can we establish a criterion on the corresponding $r$-multitableaux that guarantees that the corresponding generalized permutation is an involution?

**QUESTION 2:** Various phenomena are well known in the $S_n$-setting. For example, if we reverse the entries in the window notation of an element $\sigma$, that is, $\sigma^{\text{rev}} := \sigma_n \sigma_{n-1} \cdots \sigma_1$, then the left tableau corresponding to $\sigma$ is the transpose of the left tableau corresponding to $\sigma^{\text{rev}}$. Is there any relation to be found in the $r$-multitableaux corresponding to a generalized permutation and the $r$-multitableau corresponding to its reverse permutation?

**The Action Plan: Theoretical approach, Process, and Methodology**

The two questions above will be a good starting point for our project as they are attainable goals. These two questions in particular will help the students get quickly into the research on this topic after learning the definitions of generalized permutations in the first week of our work.

**THEORETICAL APPROACH:** The students will begin by studying the combinatorics of permutations and these generalized permutations from well-known published papers [1,5] and also my unpublished survey [3]. They will of course also read my recently published paper [2] which presents the algorithm to go from a generalized permutation to a pair of same-shape $r$-multitableaux.

**PROCESS and METHODOLOGY:** This is covered in more detail in the Mentoring Narrative for this ORSP proposal. But briefly, they will each work 6 hours each week. Two of these hours each will be with me in my office.
Dissemination Plan

Students will present their work at CERCA in Spring 2017. Also, in January 2017, if the students feel prepared by then, they will present a poster at the Joint Mathematics Meeting in Atlanta, GA. Last year, I brought my two research students to USTARS (Underrepresented Students in Topology and Algebra Research Symposium) in Texas, and I hope to again bring my current 2016-2017 research students there to present our work in April 2017.

References cited