RESEARCH NARRATIVE
ORSP Student-Faculty Research Proposal
Dr. Aba Mbirika
UWEC Department of Mathematics
July 13, 2015

**TITLE:** Lattice Point Visibility on Generalized Lines of Sights

**Motivation for the research project / Connection to previous ORSP project:**

In my 2013-2014 academic year (my first year here at UWEC as tenure-track faculty), my two research students Jasmine Nielsen and Austin Goodrich and I studied lattice point visibility on straight lines of sights from the origin. Consider the integer lattice in the plane (i.e., all the points \((x, y)\) such that \(x\) and \(y\) are integer values). We write \(\mathbb{Z}^2\) or \(\mathbb{Z} \times \mathbb{Z}\) to denote the integer lattice (where the symbol \(\mathbb{Z}\) denotes the set of integers). Imagine for a moment that each integer lattice point is an infinitely thin tree. Which trees can you see from the origin? On the \(x\)-axis, one can only see the points \((1,0)\) to the right and \((-1,0)\) to the left. And all other “trees” behind these two points are obscured from view. For example, \((2,0)\) cannot be seen because \((1,0)\) is blocking it.

See the figure on the right, where the blue vertices are examples of points which are not visible since they are obscured from view by a visible point (denoted by a tree). A natural question to ask is, what fraction of integer lattice points are visible from the origin? It turns out that this is a well-known question with an answer involving a value that is ubiquitous in mathematics, namely, the Riemann-zeta function \(\zeta(s)\) evaluated at \(s = 2\). Phrased in equivalent terms, the probability that a randomly selected lattice point is visible from the origin is \(\zeta(2)^{-1} = \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right)^{-1} = \frac{6}{\pi^2}\); that is, approximately 60%. This classic result was proven in 1883 by Cesàro [2]. In 1971 this field of lattice point visibility was revitalized by a pivotal paper by Herzog and Stewart who studied patterns in the approximate 40% of the integer lattice which is hidden from view [5]. More recently in the 1990s, papers on lattice point visibility appeared [1,7]. In 2013-2014, my research students and I used creative number theoretic techniques and tools from linear algebra to find the closest known \(4 \times 4\) and \(5 \times 5\) square pattern of invisible lattice points. In the summer of 2014, we did an SREU
funded through ORSP to write a paper and publish these results [3]. Though this paper was not completed by the end of the summer, as planned, the groundwork we laid down in this paper has been fundamental to other researchers interested in this work (in particular, with two researchers at the U.S. Military Academy at West Point – see project description below). I still plan to finish this paper and submit it to the journal *Involve* which publishes original mathematics works from undergraduate students’ collaborations with faculty.

**Project description**

The classic setting of lattice point visibility focuses on lattice points in $\mathbb{Z} \times \mathbb{Z}$ which lie on straight lines $y = ax$ through the origin with slope $a \in \mathbb{Q}$ (i.e., $a$ is a rational number). In this project we generalize this notion of lines of sights to include all curves through the origin given by power functions of the form $f(x) = ax^b$ where $a \in \mathbb{Q}$ and $b \in \mathbb{N}$ (i.e., $b$ is a positive integer). In the classic setting where $b = 1$, it is well known that the proportion of invisible lattice points is $1 - \frac{6}{\pi^2}$ (or approximately 40%). But what happens as the value $b$ increases?

In recent work in June 2015 at the U.S. Military Academy (USMA) at West Point with my two collaborators Dr. Pamela Harris of USMA and Dr. Bethany Kubik (then at USMA and now at the University of Minnesota Duluth), we gave a necessary and sufficient condition for an integer lattice point to be visible on a line of sight of the form $f(x) = ax^b$ for each fixed $b \in \mathbb{N}$. We say a lattice point $(r, s)$ is $b$-visible if it satisfies the conditions we found. One practical application being that when $b = 2$, these lines of sights are parabolas and one can think of military applications where one would want to know whether an object traveling with a parabolic trajectory hits a potential target say at an integer lattice point $(r, s)$ without hitting any other integer lattice points before reaching the target. Moreover, we proved that arbitrarily large square patterns of $b$-invisible lattice points can still be found (as is well known in the classic $b = 1$ setting). However, many questions remain open in this new setting of generalized lines of sights. In this current project with my two new UWEC research students sophomore Sara DeBrabander and junior Michele Gebert, we begin by exploring the following open questions:

1. How many points lie between $(0,0)$ and a given lattice point $(r, s)$ in $\mathbb{Z} \times \mathbb{Z}$ on the lines of sights $f(x) = ax^b$ where $a \in \mathbb{Q}$ for each fixed value $b \in \mathbb{N}$?
2. As the value \( b \) increases, does the proportion of invisible lattice points increase or decrease? A successful understanding of Question 1 may be of great assistance in answering Question 2.

3. Define a pattern \( P \) in the integer lattice to be **purely \( b \)-invisible** if all lattice points of distance less than or equal to \( \sqrt{2} \) from any point in \( P \) are \( b \)-visible. For example, on the next page we show a triangular purely \( b \)-invisible forest in white surrounded by vertices in black which are all \( b \)-visible. ([See figure on citations page 4.](#)) Can we find arbitrarily large purely \( b \)-invisible forests, shelterbelts, or other patterns?

4. The classic setting generalizes well to higher dimensions. For example, the probability that an integer lattice point \( (x_1, x_2, \ldots, x_d) \) is \( 1 \)-visible in the \( d \)-dimensional lattice \( \mathbb{Z}^d \) is \( \zeta(d)^{-1} \) where the value \( \zeta(d) \) is the Riemann zeta function \( \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \) evaluated at \( s = d \). What is the probability that a point \( (x_1, x_2, \ldots, x_d) \) in \( \mathbb{Z}^d \) is \( b \)-visible?

**The Action Plan: Theoretical approach, Process, and Methodology**

The four starting questions that I came up with are questions that I would like my undergrad research students to answer. My two West Point collaborators Drs. Pamela Harris and Bethany Kubik will develop questions for their research students to think about. My questions in particular are pitched at the levels that my two research students, having well-prepared prerequisite courses in calculus, linear algebra, and probability, can handle.

**THEORETICAL APPROACH:** The students will begin by studying the theory of lattice point visibility as laid out in both my paper with my former UWEC research students [3] and also my most recent work with my West Point collaborators [4]. They will of course also read the three papers from 1971 until the 1990s listed in this proposal [1,5,7]. The students will understand the theory of linear algebra’s intertwining with the field of number theory (a course that they have both not yet taken, but which I will teach them the necessary tools).

**PROCESS and METHODOLOGY:** This is covered in more detail in the Mentoring Narrative for this ORSP proposal. But briefly, they will each work 6 hours per week. Two of these hours will be with me in my office.

**Dissemination Plan**

Students will present their work at CERCA in Spring 2016. Since much of the previous work on generalized lines of sight is yet unpublished work with my collaborators Drs. Harris and Kubik and I, another possible avenue of dissemination is for my two research students to be co-authors with Harris, Kubik and I in a joint publication.
Figure: An example of a purely $b$-invisible triangular forest

(Note that the white vertices are $b$-invisible while black vertices are $b$-visible)

References cited

2. Cesàro, E. Question 75 (Solution). Mathesis 3 (1883), 224-225.