

On the sign representations for the complex reflection groups $G(r, p, n)$

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Abstract We present a formula for the values of the sign representations of a complex reflection group $G(r, p, n)$ in terms of its image under a generalized Robinson–Schensted algorithm.

Keywords Complex reflection groups · Robinson–Schensted map

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1 Introduction

The classical Robinson–Schensted algorithm establishes a bijection between permutations $w \in S_n$ and ordered pairs of same-shape standard Young tableaux of size n . This map has proven particularly well-suited to certain questions in the representation

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theory of both S_n and the semisimple Lie groups of type A . For instance, Kazhdan-Lusztig cells as well as the primitive spectra of semisimple Lie algebras can be readily described in terms of images of this correspondence.

Other sometimes more elementary representation-theoretic information requires more work to extract from standard Young tableaux. For instance, in independent work, Reifegerste (2004) and Sjöstrand (2005) developed a method for reading the value of the sign representation of a permutation $w \in S_n$ based on two tableaux statistics. Let $w \in S_n$ and write $RS(w) = (P, Q)$ for its image under the classical Robinson-Schensted map. If we write e for the number of squares in the even-indexed rows of P , let $sign(T)$ be the sign of a tableau T derived from its inversion number, and let sgn be the usual sign representation on S_n , then

$$sgn(w) = (-1)^e \cdot sign(P) \cdot sign(Q). \quad (1.1)$$

The focus of this note is to extend this result to the complex reflection groups $G(r, p, n)$. Its two main ingredients generalize readily to this setting. First, the classical Robinson-Schensted algorithm admits a straightforward extension mapping each element $w \in G(r, p, n)$ to a same-shape pair of r -multitableaux, see Stanley (1982, Sect. 6) and Iancu (2003). At the same time, the sign of a permutation in S_n extends to a family of r one-dimensional representations of $G(r, p, n)$. After defining new $spin$ and sgn statistics on r -multitableaux, we offer a short proof of the following:

Theorem *Let $w \in G(r, p, n)$ and write $\mathbf{RS}(w) = (\mathbf{P}, \mathbf{Q})$ for its image under the generalized Robinson-Schensted map. Given a primitive r^{th} root of unity ζ and the associated family $\{sgn_i\}_{i=0}^{r-1}$ of representations of $G(r, p, n)$, we have*

$$sgn_i(w) = (-1)^{e(\mathbf{P})} \cdot (\zeta^i)^{spin(\mathbf{P})+spin(\mathbf{Q})} \cdot sign(\mathbf{P}) \cdot sign(\mathbf{Q}),$$

where $e(\mathbf{P})$ is the total sum of the lengths of the even-indexed rows of the component tableaux of \mathbf{P} .

A weaker version of this theorem has been used to verify a formula for the sign representation of the classical Weyl groups in type B for a family of domino tableaux Robinson-Schensted maps, see (Pietraho 2014). For classical Weyl groups, all of which appear among complex reflection groups, values of the sign representation can be used to compute the Möbius function of Bruhat order (Verma 1971). The above sign formulas show that the Möbius function is well-behaved with respect to the characterization of Kazhdan-Lusztig cells by equivalence classes of tableaux and multitableaux as in Joseph (1977), Ariki (2000) and Bonnafé and Iancu (2003).

2 Preliminaries

After defining the family of complex reflection groups and their one-dimensional representations, we define multipartitions, a generalization of the Robinson-Schensted algorithm, and tableaux statistics that we will use to describe these representations.

2.1 Sign representations

Consider positive integers $r, p,$ and n with p dividing r and let ζ be the primitive root of unity $\exp(2\pi\sqrt{-1}/r)$. We define the complex reflection groups $G(r, p, n)$ as subgroups of $GL_n(\mathbb{C})$ consisting of matrices such that

- The entries are either 0 or powers of $\zeta,$
- There is exactly one nonzero entry in each row and column,
- The (r/p) -th power of the product of all nonzero entries is 1.

Together with the thirty-four exceptional groups, the groups $G(r, p, n)$ account for all finite groups generated by complex reflections (Shephard and Todd 1954), and include among them all the classical Weyl groups. In our work the parameter r will generally be fixed allowing us to write simply W_n for the group $G(r, 1, n)$. In order to establish succinct notation, we will write

$$[\zeta^{a_1}\sigma_1, \zeta^{a_2}\sigma_2, \dots, \zeta^{a_n}\sigma_n]$$

for the matrix whose nonzero entry in the i th column is ζ^{a_i} and appears in row σ_i . Utilizing this notation, define the set $S = \{s_0, \dots, s_{n-1}\}$ where

$$s_0 = [\zeta \cdot 1, 2, 3, \dots, n], \quad \text{and}$$

$$s_i = [1, 2, \dots, i - 1, i + 1, i, i + 2, \dots, n].$$

Furthermore, let $S' = \{s_0^p, s_0s_1s_0, s_i \mid 1 \leq i \leq n - 1\}$. The set S generates W_n with presentation given as

$$W_n = \left\langle s_i \mid s_0^r, s_m^2, (s_js_k)^2, (s_0s_1)^4, (s_l s_{l+1})^3, m \geq 1, |j - k| > 1, l \in [1, n - 2] \right\rangle.$$

Subject to similar relations, S' generates a subgroup $G(r, p, n)$ of W_n of index $p,$ see Ariki (1995). Let $\sigma = [\sigma_1, \dots, \sigma_n] \in S_n,$ and define $Inv(\sigma)$ to be the set of pairs (σ_i, σ_j) with $i < j$ and $\sigma_i > \sigma_j.$

There are exactly $2r$ one-dimensional representations of $W_n;$ they divide naturally into two families.

Definition 2.1 For each integer i between 0 and $r - 1,$ we define representations ζ_i and sgn_i of W_n by specifying their values on the generating set $S.$ Let

$$\tau_i^\epsilon(s_j) = \begin{cases} \zeta^i & \text{if } j = 0, \quad \text{and} \\ (-1)^\epsilon & \text{if } j = 1, \dots, n - 1 \end{cases}$$

and define $\zeta_i = \tau_i^0$ and $sgn_i = \tau_i^1.$ Each becomes a representation of the subgroup $G(r, p, n)$ by restriction.

2.2 Multitableaux

We write a partition λ of an integer m as a nonincreasing sequence of positive integers $(\lambda_1, \lambda_2, \dots, \lambda_k)$ and define its rank as $|\lambda| = m$. A *Young diagram* $[\lambda]$ of λ is a left-justified array of boxes containing λ_i boxes in its i th row. The shape of a Young diagram will refer to its underlying partition. With the integer r fixed, a *multipartition of rank n* is an r -tuple

$$\lambda = (\lambda^0, \lambda^1, \dots, \lambda^{r-1})$$

of partitions the sum of whose individual ranks equals n . The *Young diagram* $[\lambda]$ of λ is the r -tuple $([\lambda^0], \dots, [\lambda^{r-1}])$. We refer to λ as the *shape* of the diagram $[\lambda]$ and define $|\lambda| = n$. We will follow a convention of denoting objects derived from multipartitions in boldface while writing those derived from single partitions using a normal weight font.

A *standard Young tableaux of shape λ* is the Young diagram $[\lambda]$ of rank n together with a labeling of each of its boxes with the elements of $\mathbb{N}_n := \{1, 2, \dots, n\}$ in such a way that each number is used exactly once, and the labels of the boxes within each component Young diagram $[\lambda^i]$ increase along its rows and down its columns. Remembering that r is fixed, we will write \mathbf{SYT}_n for the set of all standard Young tableaux of rank n whose shape is a multipartition with r components.

Example 2.1 Take $r = 3$. The following standard Young tableau \mathbf{T} is of rank 11 and has the shape $\lambda = ((2, 1), (1, 1), (3, 3))$:

$$\mathbf{T} = \left(\begin{array}{|c|c|} \hline 6 & 11 \\ \hline 8 & \\ \hline \end{array}, \begin{array}{|c|} \hline 1 \\ \hline 7 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 2 & 4 & 9 \\ \hline 3 & 5 & 10 \\ \hline \end{array} \right)$$

Following Stanley (1982, Sect. 6) and Iancu (2003), we define a map from W_n to same-shape pairs of r -tuples of standard Young tableaux. Consider an element

$$w = [\zeta^{a_1} \sigma_1, \zeta^{a_2} \sigma_2, \dots, \zeta^{a_n} \sigma_n] \in W_n$$

and define the ordered sets $w^{(k)} = (\sigma_i \mid a_i = k)$ for $0 \leq k < r$. Let $Inv_P(w^{(k)}, w^{(l)})$ consist of $(i, j) \in Inv(\sigma)$ with $i \in w^{(l)}$ and $j \in w^{(k)}$, and let $Inv_Q(w^{(k)}, w^{(l)})$ consist of $(i, j) \in Inv(\sigma)$ with $i \in w^{(k)}$ and $j \in w^{(l)}$. Moreover, write $inv_P(w^{(k)}, w^{(l)})$ and $inv_Q(w^{(k)}, w^{(l)})$ for the respective cardinalities of these sets.

Let $RS(w^{(k)}) = (P_k, Q_k)$ be the image of the sequence $w^{(k)}$ under the classical Robinson-Schensted map, labeling squares of Q_k according to the relative positions of $i \in w^{(k)}$ within w , and define

$$\mathbf{P} := \mathbf{P}(w) = (P_0, P_1, \dots, P_{r-1}) \quad \text{and} \quad \mathbf{Q} := \mathbf{Q}(w) = (Q_0, Q_1, \dots, Q_{r-1}).$$

The multitableaux Robinson-Schensted map is defined by $\mathbf{RS}(w) = (\mathbf{P}, \mathbf{Q})$. It maps W_n onto the set of same-shape pairs of elements of \mathbf{SYT}_n and is in fact a bijection.

2.3 Tableaux and multitableaux statistics

Our goal is to describe values of the sign representations on W_n under the above generalization of the Robinson-Schensted map. To do so, we rely on a few statistics that can be readily computed from multitableaux.

Definition 2.2 An *inversion* in a Young tableau T is a pair (i, j) with $j > i$ for which the box labeled by i is contained in a row strictly below the box labeled j . Let $Inv(T)$ be the set of inversions in T , and write $inv(T)$ for its cardinality. If $\mathbf{T} = (T_0, T_1, \dots, T_{r-1})$ is a multitableau, we extend this notion and define:

$$Inv(\mathbf{T}) = \bigsqcup_k Inv(T_k) \sqcup \bigsqcup_{k < l} Inv(T_k, T_l)$$

where $Inv(T_k, T_l) = \{(j, i) \mid j > i, j \text{ is a label in } T_k, i \text{ is a label in } T_l\}$. We will be interested mainly in the parity of the size of this set and define

$$sign(\mathbf{T}) = (-1)^{inv(\mathbf{T})}.$$

Definition 2.3 For a Young tableau T , write $e(T)$ for the total number of boxes in its rows of even index. For a multitableau $\mathbf{T} = (T_0, T_1, \dots, T_{r-1})$, we write $sh(T_k)$ for the shape of the Young diagram underlying T_k and define the statistics e and $spin$ as follows:

$$e(\mathbf{T}) = \sum_{k=0}^{r-1} e(T_k) \quad \text{and} \quad spin(\mathbf{T}) = \frac{1}{2} \sum_{k=0}^{r-1} k \cdot |sh(T_k)|.$$

The $spin$ statistic provides a simple description of the image of the subgroup $G(r, p, n)$ under the r -multitableaux Robinson-Schensted map. The following is easy to verify:

Proposition 2.1 $(\mathbf{P}, \mathbf{Q}) \in \mathbf{RS}(G(r, p, n))$ if and only if $2 spin(\mathbf{P}) \equiv 0 \pmod{p}$.

2.4 A set of functions and an example

We define a family of functions on W_n . In the next section we will show that they coincide with the sign representations on W_n . Again, for $w \in W_n$, let $\mathbf{RS}(w) = (\mathbf{P}, \mathbf{Q})$. For $0 \leq i < r$, we will write

$$\pi_i(w) = (-1)^{e(\mathbf{P})} \cdot (\zeta^i)^{spin(\mathbf{P})+spin(\mathbf{Q})} \cdot sign(\mathbf{P}) \cdot sign(\mathbf{Q}).$$

Example 2.2 Consider $w = [\zeta^1 5, 1, \zeta^2 3, 6, \zeta^2 7, \zeta^1 4, 2, 8]$ in $G(4, 1, 8)$. Recalling the notation in Sect. 2.2, we have $w^{(0)} = (1, 6, 2, 8)$, $w^{(1)} = (5, 4)$, $w^{(2)} = (3, 7)$, and $w^{(3)} = \emptyset$. Furthermore,

$$\begin{aligned}
 RS(w^{(0)}) &= \left(\begin{array}{|c|c|c|} \hline 1 & 2 & 8 \\ \hline 6 & & \end{array}, \begin{array}{|c|c|c|} \hline 2 & 4 & 8 \\ \hline 7 & & \end{array} \right) & RS(w^{(1)}) &= \left(\begin{array}{|c|} \hline 4 \\ \hline 5 \\ \hline \end{array}, \begin{array}{|c|} \hline 1 \\ \hline 6 \\ \hline \end{array} \right) \\
 RS(w^{(2)}) &= \left(\begin{array}{|c|} \hline 3 \\ \hline 7 \\ \hline \end{array}, \begin{array}{|c|} \hline 3 \\ \hline 5 \\ \hline \end{array} \right) & RS(w^{(3)}) &= (\emptyset, \emptyset).
 \end{aligned}$$

From these we construct the Robinson-Schensted image of w :

$$\mathbf{RS}(w) = (\mathbf{P}, \mathbf{Q}) = \left(\left(\begin{array}{|c|c|c|} \hline 1 & 2 & 8 \\ \hline 6 & & \end{array}, \begin{array}{|c|} \hline 4 \\ \hline 5 \\ \hline \end{array}, \begin{array}{|c|} \hline 3 \\ \hline 7 \\ \hline \end{array}, \emptyset \right), \left(\begin{array}{|c|c|c|} \hline 2 & 4 & 8 \\ \hline 7 & & \end{array}, \begin{array}{|c|} \hline 1 \\ \hline 6 \\ \hline \end{array}, \begin{array}{|c|} \hline 3 \\ \hline 5 \\ \hline \end{array}, \emptyset \right) \right).$$

We read off $inv(\mathbf{P}) = 10$, $inv(\mathbf{Q}) = 14$, $e(\mathbf{P}) = 2$, and $spin(\mathbf{P}) = spin(\mathbf{Q}) = 3$. Hence $\pi_i(w) = (\zeta^i)^2$ which coincides with $sgn_i(w)$.

3 Sign under the Robinson-Schensted map

With the appropriate definitions of the tableaux statistics in place, we can now verify the claimed formulas for the family of sign representations $\{sgn_i\}$. Recall our notation $w = [\zeta^{a_1}\sigma_1, \zeta^{a_2}\sigma_2, \dots, \zeta^{a_n}\sigma_n] \in W_n$ where $\sigma = \sigma_1 \dots \sigma_n \in S_n$. Directly from the definitions of Inv_P and Inv_Q , we obtain the following partition:

$$Inv(\sigma) = \bigsqcup_{k=0}^{r-1} Inv(w^{(k)}) \bigsqcup_{k<l} Inv_P(w^{(k)}, w^{(l)}) \bigsqcup_{k<l} Inv_Q(w^{(k)}, w^{(l)}). \tag{3.1}$$

Applying this to sgn_i , we have:

$$sgn_i(w) = (\zeta^i)^{\sum_{k=1}^n a_k} \cdot \prod_{k=0}^{r-1} sgn(w^{(k)}) \cdot \prod_{k<l} (-1)^{inv_P(w^{(k)}, w^{(l)}) + inv_Q(w^{(k)}, w^{(l)})}. \tag{3.2}$$

Write the reverse of $\sigma \in S_n$ as $\sigma^{rev} := \sigma_n \sigma_{n-1} \dots \sigma_1 \in S_n$ and for fixed integers k, l such that $0 \leq k < l \leq r - 1$, let

$$\begin{aligned}
 \mathcal{I}_1 &= Inv_P(w^{(k)}, w^{(l)}) = \left\{ (i, j) \in Inv(\sigma) \mid i \in w^{(l)} \text{ and } j \in w^{(k)} \right\}, \\
 \mathcal{I}_2 &= Inv_Q(w^{(k)}, w^{(l)}) = \left\{ (i, j) \in Inv(\sigma) \mid i \in w^{(k)} \text{ and } j \in w^{(l)} \right\}, \text{ and} \\
 \mathcal{I}_3 &= \left\{ (i, j) \in Inv(\sigma^{rev}) \mid i \in w^{(k)} \text{ and } j \in w^{(l)} \right\}.
 \end{aligned}$$

Lemma 3.1 *For fixed integers k and l as above, we have*

$$Inv(P_k, P_l) = \mathcal{I}_2 \sqcup \mathcal{I}_3 \text{ and } |Inv(Q_k, Q_l)| = |\mathcal{I}_1 \sqcup \mathcal{I}_3|.$$

Proof To prove the first claim, let $(\sigma_i, \sigma_j) \in Inv(P_k, P_l)$. Then $\sigma_i > \sigma_j$, $\sigma_i \in w^{(k)}$, and $\sigma_j \in w^{(l)}$. If $i < j$ then $(\sigma_i, \sigma_j) \in Inv(\sigma)$ and hence lies in \mathcal{I}_2 . On the other

hand if $i > j$, then $(\sigma_i, \sigma_j) \in \text{Inv}(\sigma^{\text{rev}})$ and hence lies in \mathcal{S}_3 . To prove the second claim, let $(j, i) \in \text{Inv}(Q_k, Q_l)$. Then $j > i$, $\sigma_j \in w^{(k)}$, and $\sigma_i \in w^{(l)}$. If $\sigma_i > \sigma_j$, then $(\sigma_i, \sigma_j) \in \text{Inv}(\sigma)$ and hence lies in \mathcal{S}_1 . On the other hand if $\sigma_i < \sigma_j$, then $(\sigma_j, \sigma_i) \in \text{Inv}(\sigma^{\text{rev}})$ and hence lies in \mathcal{S}_3 . Thus each $(j, i) \in \text{Inv}(Q_k, Q_l)$ corresponds to either a $(\sigma_i, \sigma_j) \in \mathcal{S}_1$ or a $(\sigma_j, \sigma_i) \in \mathcal{S}_3$.

Immediately, we obtain:

Corollary 3.1 *For any $k < l$,*

$$\text{inv}_P(w^{(k)}, w^{(l)}) + \text{inv}_Q(w^{(k)}, w^{(l)}) \equiv \text{inv}(P_k, P_l) + \text{inv}(Q_k, Q_l) \pmod{2}.$$

We are now ready to prove that the functions π_i defined in Sect. 2.4 coincide with sgn_i , hence proving our main theorem.

Theorem 3.1 *Let $w \in W_n$ and write $\mathbf{RS}(w) = (\mathbf{P}, \mathbf{Q})$ for its image under the generalized Robinson-Schensted map. Given a primitive r^{th} root of unity ζ and the associated family $\{\text{sgn}_i\}_{i=0}^{r-1}$ of representations of W_n , we have*

$$\text{sgn}_i(w) = (-1)^{e(\mathbf{P})} \cdot (\zeta^i)^{\text{spin}(\mathbf{P})+\text{spin}(\mathbf{Q})} \cdot \text{sign}(\mathbf{P}) \cdot \text{sign}(\mathbf{Q}).$$

Proof Observe that the functions π_i can be decomposed as follows

$$\begin{aligned} \pi_i(w) &= (-1)^{e(\mathbf{P})} \cdot (\zeta^i)^{\text{spin}(\mathbf{P})+\text{spin}(\mathbf{Q})} \cdot \text{sign}(\mathbf{P}) \cdot \text{sign}(\mathbf{Q}) \\ &= (-1)^{\sum_{k=0}^{r-1} e(P_k)} (\zeta^i)^{\sum_{k=1}^n a_k} \prod_{k=0}^{r-1} \text{sign}(P_k) \prod_{k=0}^{r-1} \text{sign}(Q_k) \prod_{k < l} (-1)^{\text{inv}(P_k, P_l) + \text{inv}(Q_k, Q_l)} \\ &= (\zeta^i)^{\sum_{k=1}^n a_k} \cdot \prod_{k=0}^{r-1} \left((-1)^{e(P_k)} \text{sign}(P_k) \text{sign}(Q_k) \right) \cdot \prod_{k < l} (-1)^{\text{inv}(P_k, P_l) + \text{inv}(Q_k, Q_l)} \end{aligned}$$

Since each $(-1)^{e(P_k)} \text{sign}(P_k) \text{sign}(Q_k)$ coincides with $\text{sgn}(w^{(k)})$ by Eq. (1.1), we have

$$\begin{aligned} \pi_i(w) &= (\zeta^i)^{\sum_{k=1}^n a_k} \cdot \prod_{k=0}^{r-1} \text{sgn}(w^{(k)}) \cdot \prod_{k < l} (-1)^{\text{inv}(P_k, P_l) + \text{inv}(Q_k, Q_l)} \\ &= (\zeta^i)^{\sum_{k=1}^n a_k} \cdot \prod_{k=0}^{r-1} \text{sgn}(w^{(k)}) \cdot \prod_{k < l} (-1)^{\text{inv}_P(w^{(k)}, w^{(l)}) + \text{inv}_Q(w^{(k)}, w^{(l)})} \\ &= \text{sgn}_i(w) \end{aligned}$$

where the second equality holds as a consequence of Corollary 3.1 and the final equality holds by Eq. (3.2).

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