“Mathematics knows no races or geographic boundaries; for mathematics, the cultural world is one country.”
– David Hilbert (1862-1943)

Course Packet written by aBa Mbirika
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¹Since the final exam is a cumulative exam, the three study packets are a great review for the final exam.
Preface: Finite Mathematics is a course covering a large variety of different subjects. No mathematics beyond high school algebra or the equivalent of Math 20 at UWEC is required. The textbook and this course packet are essential to the course. You will be expected to read the textbook on your own before each class, especially on the section which we will cover that day. Together in class we will be creating the missing contents of this course packet. In the words of mathematician Paul Halmos (1916–2006):

“The only way to learn mathematics is to do mathematics.”

To that end, we will have three to four homework assignments each week on WeBWorK (an online homework platform).

Acknowledgments: I would like to express my sincere appreciation to my many former Math 104 students who helped me sculpt and perfect these course notes over the years. I also would like to express my thanks to the following mathematicians\(^2\) for their useful comments and suggestion in earlier drafts of this course pack:

(1) Niels Abel 
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(34) Bernhard Riemann 
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\(^2\)aBa does indeed realize that every person on his list of acknowledgments is deceased. But that mortality status did not prevent any of these 35 people from giving thoughtful comments and spiritual inspiration to him in the writing of this course pack!
About your Instructor: aBa was born and raised in New York City. He completed his bachelors degree from Sonoma State University in California. After receiving his PhD from the University of Iowa, he then completed a 3-year postdoctoral position at Bowdoin College in Maine. He is now an associate professor at the University of Wisconsin-Eau Claire (UWEC). His mathematical interests include combinatorics as related to areas such as representation theory and algebraic geometry/topology. His research interests with UWEC undergraduates have included lattice point visibility, complex reflection groups, the Fibonacci sequence modulo $m$, and number theory in the Eisenstein integer ring.

The triangle is the Greek letter $\Delta$, which is delta, and in mathematics it signifies “change”. Math 104 will CHANGE your life for the better!!

About your Teaching Assistant: Bec Blaedow is from Sussex, Wisconsin. She is going for her Bachelors of Music Education-Choral with an Adaptive Music Certificate to be certified to teach students with special needs. She hopes to teach middle school! She loves swimming, rock climbing, biking, singing, playing piano and guitar, spending time with friends, and grading quizzes and exams with aBa! Bec enjoys listening to music and helping out in the theatre department! She has made costumes for many UWEC Productions and was the lead make up designer for “James and the Giant Peach” last spring! Her passion is teaching and giving advise to our freshly hatched BluGolds!

About your SI-Leader: Franny Eckendorf (affectionately called Fran Mother 😊 in our class) is your Supplemental Instruction (SI) Leader. She’s from Stevens Point, Wisconsin, and is an Elementary Education major. Some little things about her are that she loves to sing, dance, and do theatre. Fran Mother is so excited to have a great, successful semester with y’all.
1.1 Coordinate Systems and Graphs

QUESTION: Who is the person in the image below?

HINT: “Cogito ergo sum.”

Came to Earth on: March 31st, 1596 (La Haye en Touraine, Kingdom of France)
Departed Earth on: Feb 11th, 1650 (Stockholm, Swedish Empire)

Mathematical Legacy:

- He invented the convention of using variables $x, y,$ and $z$.
- He pioneered the standard notation of using superscripts for exponents.
- He was the first to profess that an abstract quantity such as $a^2$ can represent length as well as area. Formerly, only something like $a^2$ was only considered geometric.
- His “rule of signs” is still the common method to determine the number of positive and negative roots of a polynomial.
- In physics, he was the first to understand the following: When you see a rainbow, you are actually seeing light that bounce off a rain droplet and reflects back to your eye at an angle of $42^\circ$. 
HIS MOST WELL-KNOWN CONTRIBUTION TO MATHEMATICS:

Two Algebraic Representations of a Line:

- **(Standard Form)** \( y = mx + b \) where \( m \) and \( b \) are real numbers
- **(General Form)** \( cx + dy = e \) where \( c, d, \) and \( e \) are real numbers

**QUESTIONS:** Convert the following into standard form.

- \( 14x + 7y = 21 \)
- \( -\frac{1}{2}x + \frac{2}{3}y = 10 \)

**QUESTIONS:** Are the points (3,5) and (5,17) on the line \( 8x - 4y = 4 \)?
How to Graph a Line:

1. First put it in standard form $y = mx + b$.
2. Plot the $y$-intercept $(0, b)$.
3. Plot some other point. Usually we pick the $x$-intercept. **HINT:** Set $y = 0$ and solve for $x$.
4. Draw a line between the two points.

**QUESTION:** Graph the line $\frac{1}{2}x - \frac{1}{3}y = -1$.

**QUESTION:** The temperature of water in a heating tea kettle rises according to the equation $y = 30x + 72$, where $y$ is the temperature (in degrees Fahrenheit) $x$ minutes after the kettle was put on the burner.

(a) What physical interpretation can be given to the $y$-intercept of the graph?

(b) What will the temperature of the water be after 3 minutes?

(c) After how many minutes will the water be at its boiling point of 212°?
45. College Freshmen The percent of college freshmen who enter college intending to major in general biology has increased steadily in recent years. The percent, \( y \), who entered college \( x \) years after 2000 intending to major in general biology is approximated by the linear equation \( y = .2x + 3.9 \). (Source: The American Freshman: National Norms.)

(a) What interpretation can be given to the \( y \)-intercept of the graph of the equation?

(b) Approximately what percent of college freshmen entering in 2011 intended to major in general biology? How does this compare with the actual value, 6.2%?

(c) In what year did approximately 5.3% of entering college freshmen intend to major in general biology?

46. College Freshmen The percent of college freshmen who smoke tobacco products has decreased steadily in recent years. The percent, \( y \), who smoked \( x \) years after 2000 is approximated by the linear equation \( y = -.65x + 9.7 \). (Source: The American Freshman: National Norms.)

(a) What interpretation can be given to the \( y \)-intercept of the graph of the equation?

(b) Approximately what percent of college freshmen smoked in 2004?

(c) In what year did approximately 4.5% of college freshmen smoke?
1.2 Linear Inequalities

QUESTION: How is an alligator like an inequality?

THREE PROPERTIES:

I. If $a < b$ and $c$ is any number, then $[\text{what?}]$

\[
\begin{align*}
    a + c & \quad b + c \\
    a - c & \quad b - c.
\end{align*}
\]

II. If $a < b$ and $c$ is positive, then $[\text{what?}]

\[
\begin{align*}
    ac & \quad bc.
\end{align*}
\]

III. If $a < b$ and $c$ is negative, then $[\text{what?}]

\[
\begin{align*}
    ac & \quad bc.
\end{align*}
\]

SOLVE THE INEQUALITY: $-5x + 13 \leq -2$
DEFINITION: A linear inequality is an inequality of the form
\[ cx + dy \leq e \quad \text{or} \quad cx + dy \geq e \]
where \( c, d, \) and \( e \) are real numbers. The standard forms are the following:
\[ y \leq mx + b \quad \text{or} \quad y \geq mx + b. \]

QUESTION: Put the inequality \( 5x - \frac{1}{3}y \leq 6 \) into standard form.

GRAPHING LINEAR INEQUALITIES:
Generally, we shade the “bad part” and what is left unshaded is the solution to a linear inequality.

- Graph \( x \geq 5 \) on the graph below.
- Graph \( y \leq -2 \) on the graph below.
**DEFINITION:** The solution set of a linear inequality is called the **feasible set**. The feasible set contains the points \((a, b)\) which satisfy the linear inequality.

**USE THE GRAPHS YOU DREW ON THE PREVIOUS PAGE TO ANSWER THE FOLLOWING:**

- **[TRUE or FALSE]** The point \((4, -3)\) is in the feasible set of \(x \geq 5\) ?

- **[TRUE or FALSE]** The point \((4, -3)\) is in the feasible set of \(y \leq -2\) ?

**GRAPHING SYSTEMS OF LINEAR INEQUALITIES:** To graph a system of linear inequalities, you look to see where the feasible sets **OVERLAP**.

**HINT:** Shade the “bad stuff” in both and see what remains unshaded!

\[
\begin{align*}
y &\leq -2 \\
4x - 9y &\leq 36
\end{align*}
\]

**TIP:** First put \(4x - 9y \leq 36\) into standard form! Then find the \(x\)- and \(y\)-intercepts and draw the line \(y = mx + b\) and shade the appropriate side.
A Problem Similar to Last Problem In WeBWorK 2.1
Homework Problem #7

**NOTE:** In this problem, the green shaded area is the feasible set.

**QUESTION:**
Find the three inequalities that describe the feasible set. And write them in general form (with the coefficient of $x$ having a positive value.)
Group Work on Ch.1.3

1. Suppose that the demand curve for soybeans has the equation $p = -2.2q + 19.36$ and the supply curve for soybeans has the equation $p = 1.5q + 9$ where $p$ is the price per bushel in dollars and $q$ is the quantity (demanded or produced) in billions of bushels.

   a) Find the quantities supplied and demanded when the price of soybeans is 16.50 per bushel.

   b) Determine the quantity of soybeans that will be produced and the quantity at which it will sell (i.e., the equilibrium point).

2. The admission fee at an amusement park is 1.5 dollars for children and 4 dollars for adults. On a certain day, 349 people entered the park, and the admission fees collected totaled 936 dollars. How many children and how many adults were admitted?

   **HINT:** Let $x = \text{number of children}$, $y = \text{number of adults}$, then create two linear equations. For example, since there are 349 people in the park, one such equation would be $x + y = 349$. Now create a 2nd equation using the fact that 936 dollars was collected. After you have two equations, SOLVE this system of linear equations.
3. Graph the feasible set for the system of linear inequalities:

\[
\begin{align*}
    x + 4y & \leq 28 \\
    x + y & \leq 10 \\
    3x + y & \leq 24 \\
    x & \geq 0, \quad y \geq 0
\end{align*}
\]

In your graph label the intersection points of all the lines (including if they intersect the x-axis or y-axis). [See solution to problem 19 in book on pg. 21 for a VERY similar example.]

**HINT:** First write the first three inequalities in standard form (i.e., \(y \leq mx + b\) or \(y \geq mx + b\)), and then shade the region is NOT part of the feasible set. In the end, the remaining part of the graph that is NOT shaded is called the feasible set. This is the way it is done in the book – but not in that one WeBWorK problem, so far, that we had (where the feasible set was the shaded part).
32. Cost Equation Suppose that the total cost $y$ of making $x$ coats is given by the formula $y = 40x + 2400$.
   (a) What is the cost of making 100 coats?
   (b) How many coats can be made for $3600$?
   (c) Find and interpret the $y$-intercept of the graph of the equation.
   (d) Find and interpret the slope of the graph of the equation.

33. Revenue Equation Suppose that the total revenue $y$ from the sale of $x$ coats is given by the formula $y = 100x$.
   (a) What is the revenue if 300 coats are sold?
   (b) How many coats must be sold to have a revenue of $6000$?
   (c) Find and interpret the $y$-intercept of the graph of the equation.
   (d) Find and interpret the slope of the graph of the equation.

34. Profit Equation Consider a coat factory with the cost and revenue equations given in Exercises 32 and 33.
   (a) Find the equation giving the profit $y$ resulting from making and selling $x$ coats.
   (b) Find and interpret the $y$-intercept of the graph of the equation.
   (c) Find and interpret the $x$-intercept of the graph of the equation.
   (d) Find and interpret the slope of the graph of the equation.
   (e) How much profit will be made if 80 coats are sold?
   (f) How many coats must be sold to have a profit of $6000$?
   (g) Sketch the graph of the equation found in part (a).
Is This Page Intentionally Left Blank for Your Doodling Delight?

Nope. It is placed here to ensure that the new section on the next page begins with an odd-numbered page.
2.1 Solving Systems of Linear Eqns.

Solve \[ \begin{align*}
2x - 6y &= -8 \\
-5x + 13y &= 1
\end{align*} \]

using Gauss-Jordan elimination.

\[ \begin{align*}
2x - 6y &= -5 \quad \overset{\text{new}}{\Rightarrow} \quad \begin{align*}
\frac{1}{2}R_1 &\rightarrow R_1 \\
x - 3y &= -4
\end{align*}
\end{align*} \]

\[ \begin{align*}
5R_1 + R_2 &\overset{\text{new}}{\rightarrow} R_2 \\
x - 3y &= -4 \\
-2y &= -19
\end{align*} \]

\[ \begin{align*}
-\frac{1}{2}R_2 &\rightarrow R_2 \\
x - 3y &= -4 \\
y &= \frac{19}{2}
\end{align*} \]

\[ \begin{align*}
3R_2 + R_1 &\rightarrow R_1 \\
x &= \frac{49}{2} \\
y &= \frac{19}{2}
\end{align*} \]

Now get rid of the variables & write matrix:

\[ \begin{bmatrix}
2 & -6 & -8 \\
-5 & 13 & 1
\end{bmatrix} \overset{\text{new}}{\Rightarrow} \begin{bmatrix}
1 & -3 & -4 \\
-5 & 13 & 1
\end{bmatrix} \]

etc
Use Gauss Jordan Elimination to solve the system:

\[
\begin{align*}
    x - 2y + z &= 0 \\
y - 2z &= 4 \\
x + y + 3z &= 5
\end{align*}
\]

\[
\begin{align*}
    -4R_1 + R_3 &\rightarrow \text{new } R_3 \\
x - 2y + z &= 0 \\
y - 2z &= 4 \\
9y - z &= 5
\end{align*}
\]

\[
\begin{align*}
    -9R_2 + R_3 &\rightarrow \text{new } R_3 \\
x - 2y + z &= 0 \\
y - 2z &= 4 \\
17z &= -31
\end{align*}
\]

\[
\begin{align*}
    2R_2 + R_1 &\rightarrow \text{new } R_1 \\
x &= 8 \\
y - 2z &= 4 \\
17z &= -31
\end{align*}
\]

\[
\begin{align*}
    \frac{1}{17}R_2 &\rightarrow \text{new } R_3 \\
x &= 8 \\
y - 2z &= 4 \\
z &= -31/17
\end{align*}
\]

\[
\begin{align*}
    2R_3 + R_2 &\rightarrow \text{new } R_2 \\
&\quad \text{and } 3R_2 + R_1 \rightarrow \text{new } R_1 \\
x &= 43/17 \\
y &= 6/17 \\
z &= -31/17
\end{align*}
\]
Example 4 (Sec. 2.1)

Use Gauss-Jordan Elimination to solve \( \begin{cases} 2x - 6y = -8 \\ -5x + 13y = 1 \end{cases} \)

Rewrite in Matrix Form
(i.e. "augmented matrix" or initial matrix)

\[
\begin{bmatrix}
2 & -6 & | & -8 \\
-5 & 13 & | & 1
\end{bmatrix}
\]

Get a 1 in the top left by doing \( \frac{1}{2} R_1 \rightarrow \text{new } R_1 \)

\[
\begin{bmatrix}
1 & -3 & | & -4 \\
-5 & 13 & | & 1
\end{bmatrix}
\]

5R₁ + R₂ \rightarrow new R₂

\[
\begin{bmatrix}
1 & -3 & | & -4 \\
0 & -2 & | & -19
\end{bmatrix}
\]

\((-\frac{1}{2})R₂ \rightarrow \text{new } R₂\)

\[
\begin{bmatrix}
1 & -3 & | & -4 \\
0 & 1 & | & 19\frac{1}{2}
\end{bmatrix}
\]

3R₂ + R₁ \rightarrow new R₁

\[
\begin{bmatrix}
1 & 0 & | & 49\frac{1}{2} \\
0 & 1 & | & 19\frac{1}{2}
\end{bmatrix}
\]

Translate last matrix back into math.

\[
\begin{cases}
x + 0y = 49\frac{1}{2} \\
0x + 1y = 19\frac{1}{2}
\end{cases}
\]

So \( \begin{cases} x = 49\frac{1}{2} \\ y = 19\frac{1}{2} \end{cases} \) Answer is \( \left( \frac{49}{2}, \frac{19}{2} \right) \)
2.1 & 2.2

The Method (Gauss-Jordan Elimination)

1. Rewrite system in Matrix Form (i.e. the "augmented matrix", or the initial matrix)

2. If there is a 1 in column 1, exchange rows if necessary to put 1 in upper left.
   (by Row Operation 1)

3. Get a 1 in top of column 1 (by Row Operation 2)

4. Produce 0's below this to make
   (by Row Operation 3)
   \[
   \begin{bmatrix}
   1 & 0 & 0 & 0 \\
   0 & 0 & 0 & 0 \\
   0 & 0 & 0 & 0 \\
   \end{bmatrix}
   \]
   \[K_2 + R_2 \rightarrow \text{new } R_i\]

5. Get a 1 in 2nd row of 2nd column (by Row Operation 2)
   Then produce 0's below & above it to make
   (by Row Operation 3)
   \[
   \begin{bmatrix}
   1 & 0 & 0 & 0 \\
   0 & 0 & 0 & 0 \\
   0 & 0 & 0 & 0 \\
   \end{bmatrix}
   \]
   \[K_2 + R_2 \rightarrow \text{new } R_i\]

6. Get a 1 in 3rd row of 3rd column (by Row Operation 2)
   Then produce 0's above it to make
   (by Row Operation 3)
   \[
   \begin{bmatrix}
   1 & 0 & 0 & a \\
   0 & 0 & 0 & b \\
   0 & 0 & 0 & c \\
   \end{bmatrix}
   \] (diagonal form!!)
   \[
   \begin{bmatrix}
   K_2 + R_2 \rightarrow \text{new } R_i
   \end{bmatrix}
   \]

7. The only solution is \((x, y, z)\), or \[x = a, \quad y = b, \quad z = c\]
Three Types of Possibilities:

Example 5 (in Section 2.1)

\[
\begin{align*}
\begin{cases}
3x - 6y + 9z &= 0 \\
4x - 6y + 8z &= -4 \\
-2x - y + z &= 7
\end{cases}
\Rightarrow
\begin{bmatrix}
3 & -6 & 9 & 0 \\
4 & -6 & 8 & -4 \\
-2 & -1 & 1 & 7
\end{bmatrix}
\xrightarrow{\text{many moves later}}
\begin{bmatrix}
1 & 0 & 0 & -3 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1
\end{bmatrix}
\]

Convert the last matrix back into equations & you get

\[
\begin{align*}
\begin{cases}
x + oy + oz &= -3 \\
0x + y + oz &= 0 \\
0x + oy + z &= 1
\end{cases}
\Rightarrow
\begin{cases}
x = -3 \\
y = 0 \\
z = 1
\end{cases}
\]

Example 2 (in Section 2.2)

\[
\begin{align*}
\begin{cases}
x - y + z &= 3 \\
x + y - z &= 5 \\
-2x + 4y - 4z &= 1
\end{cases}
\Rightarrow
\begin{bmatrix}
1 & -1 & 1 & 3 \\
1 & 1 & -1 & 5 \\
-2 & 4 & -4 & 1
\end{bmatrix}
\xrightarrow{\text{5 moves later}}
\begin{bmatrix}
1 & 0 & 0 & 4 \\
0 & 1 & -1 & 1 \\
0 & 0 & 0 & 5
\end{bmatrix}
\]

Convert last matrix back to math

\[
\begin{align*}
\begin{cases}
x &= 4 \\
y - z &= 1 \\
o &= 5
\end{cases}
\Rightarrow \text{No Solution} \quad \text{Huh!!}
\]

Example 3 (in Section 2.2)

\[
\begin{align*}
\begin{cases}
2x + 2y + 4z &= 8 \\
x - y + 2z &= 2 \\
-x + 5y - 2z &= 2
\end{cases}
\Rightarrow
\begin{bmatrix}
2 & 2 & 4 & 8 \\
1 & -1 & 2 & 2 \\
-1 & 5 & -2 & 2
\end{bmatrix}
\xrightarrow{\text{6 moves later}}
\begin{bmatrix}
1 & 0 & 2 & 3 \\
0 & 1 & 0 & 1 \\
0 & 0 & 0 & 6
\end{bmatrix}
\]

Convert last matrix back to math

\[
\begin{align*}
\begin{cases}
x + 2z &= 3 \\
y &= 1 \\
o &= 0
\end{cases}
\Rightarrow \text{There are two ways to present the infinitely many solutions.}
\]
One Way: \( ( \text{Take } x + 2z = 3 \text{ & Solve for } z ) \)
\[
\begin{align*}
  x &= x \\
  y &= 1 \\
  z &= -\frac{1}{2}x + \frac{3}{2}
\end{align*}
\]

So \( 2z = -x + 3 \)

Thus \( z = -\frac{1}{2}x + \frac{3}{2} \)

For example, if \( x = 0 \), then \((0,1,\frac{3}{2})\) is one of infinitely many solutions.

The Other Way: \( ( \text{Take } x + 2z = 3 \text{ & Solve for } x ) \)
\[
\begin{align*}
  x &= -2z + 3 \\
  y &= 1 \\
  z &= z
\end{align*}
\]

For example, if \( z = 0 \), then \((3,1,0)\) is one of infinitely many solutions.

Observe that if \( z = \frac{3}{2} \), then we recover the \((0,1,\frac{3}{2})\) solution which we got above.
2.2 General Systems of Linear Equations

**QUESTION:** What are the four steps to transform a matrix from

\[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix} * \]

into the diagonal form

\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} *
\]

**ANSWER:** Do the following

\[
\begin{bmatrix}
\text{Step 1: Get a 1 here} & \text{Step 4: Get a 0 here} \\
\text{Step 2: Get a 0 here} & \text{Step 3: Get a 1 here}
\end{bmatrix} *
\]

---

**QUESTION:** What are the nine steps to transform a matrix from

\[
\begin{bmatrix}
a & b & c \\
d & e & f \\
g & h & i
\end{bmatrix} * \]

into the diagonal form

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} *
\]

**ANSWER:** Do the following

\[
\begin{bmatrix}
\text{Step 1: Get a 1 here} & \text{Step 6: Get a 0 here} & \text{Step 9: Get a 0 here} \\
\text{Step 2: Get a 0 here} & \text{Step 4: Get a 1 here} & \text{Step 8: Get a 0 here} \\
\text{Step 3: Get a 0 here} & \text{Step 5: Get a 0 here} & \text{Step 7: Get a 1 here}
\end{bmatrix} *
\]

---

**Recall the 3 Legal Row Operations:**

**Row Operation 1)** Switch two rows.

\[ R_i \leftrightarrow R_j \text{ (This is ONLY done, when possible, for Step 1).} \]

**Row Operation 2)** Multiply a row by a number and replace that row.

\[ k \cdot R_i \rightarrow \text{new } R_i \text{ (This is how you get a 1).} \]

**Row Operation 3)** Add a multiple of one row to another row to change that other row.

\[ k \cdot R_i + R_j \rightarrow \text{new } R_j \text{ (This is how you get a 0).} \]
**Example of Unique Solution**

**EXERCISE:** Solve the following system using Gauss-Jordan row elimination.

\[
\begin{align*}
3x + 15y &= 21 \\
-2x - 7y &= -5
\end{align*}
\]

**STEP 0)** Create the augmented matrix (also called the initial matrix). That is, translate the math into matrix language!

\[
\begin{bmatrix}
\end{bmatrix}
\]

<table>
<thead>
<tr>
<th>Step #</th>
<th>Row Operation</th>
<th>New Matrix</th>
<th>Scratchwork</th>
</tr>
</thead>
</table>
| Step 1 |               | \begin{bmatrix}
\end{bmatrix} |             |
| Step 2 |               | \begin{bmatrix}
\end{bmatrix} |             |
| Step 3 |               | \begin{bmatrix}
\end{bmatrix} |             |
| Step 4 |               | \begin{bmatrix}
\end{bmatrix} |             |

**Final Step)** Translate the last matrix back into math language!
**Example of No Solution**

**EXERCISE:** Solve the following system using Gauss-Jordan row elimination.

\[
\begin{align*}
-x + 3y &= 11 \\
3x - 9y &= -30 \\
\end{align*}
\]

**STEP 0** Create the augmented matrix (also called the initial matrix). That is, translate the math into matrix language!

\[
\begin{bmatrix}
\end{bmatrix}
\]

<table>
<thead>
<tr>
<th>Step #</th>
<th>Row Operation</th>
<th>New Matrix</th>
<th>Scratchwork</th>
</tr>
</thead>
</table>
| Step 1 |               | \[
\begin{bmatrix}
\end{bmatrix}
\] |             |
| Step 2 |               | \[
\begin{bmatrix}
\end{bmatrix}
\] |             |
| Step 3 |               | \[
\begin{bmatrix}
\end{bmatrix}
\] |             |
| Step 4 |               | \[
\begin{bmatrix}
\end{bmatrix}
\] |             |

**Final Step** Translate the last matrix back into math language!
Example of Infinitely Many Solutions

**EXERCISE:** Solve the following system using Gauss-Jordan row elimination.

\[
\begin{align*}
2x - 4y &= 6 \\
-x + 2y &= -3
\end{align*}
\]

**STEP 0** Create the augmented matrix (also called the initial matrix). That is, translate the math into matrix language!

\[
\begin{bmatrix}
2 & -4 & | & 6 \\
-1 & 2 & | & -3
\end{bmatrix}
\]

<table>
<thead>
<tr>
<th>Step #</th>
<th>Row Operation</th>
<th>New Matrix</th>
<th>Scratchwork</th>
</tr>
</thead>
</table>
| Step 1 |               | \[
\begin{bmatrix}
2 & -4 & | & 6 \\
-1 & 2 & | & -3
\end{bmatrix}
\] |             |
| Step 2 |               | \[
\begin{bmatrix}
2 & -4 & | & 6 \\
0 & 0 & | & 0
\end{bmatrix}
\] |             |
| Step 3 |               | \[
\begin{bmatrix}
1 & 0 & | & 3 \\
0 & 0 & | & 0
\end{bmatrix}
\] |             |
| Step 4 |               | \[
\begin{bmatrix}
1 & 0 & | & 3 \\
0 & 1 & | & 0
\end{bmatrix}
\] |             |

**Final Step** Translate the last matrix back into math language!
RECALL: What are the three possibilities on how two lines in the plane can interact? Draw them below.

QUESTION: What the three ways that three planes in the 3-space can interact? That is, what are the possibilities for the answer to the system?

\[
\begin{align*}
ax + by + cz &= *_1 \\
 dx + ey + fz &= *_2 \\
gx + hy + iz &= *_3
\end{align*}
\]
QUESTION: If the following system has matrix that reduces to the following, then what is the solution to the system?

\[
\begin{align*}
    x - y + 3z &= 3 \\
    -2x + 3y - 11z &= -4 \\
    x - 2y + 8z &= 6
\end{align*}
\]

\[
\begin{bmatrix}
    1 & -1 & 3 & 3 \\
    -2 & 3 & -11 & -4 \\
    1 & -2 & 8 & 6
\end{bmatrix}
\]

Then, many moves later this can become the following matrix:

\[
\begin{bmatrix}
    1 & 0 & -2 & 5 \\
    0 & 1 & -5 & 2 \\
    0 & 0 & 0 & 5
\end{bmatrix}
\]

That is, step 7 cannot be done since there is no way to make the zero in the 3\textsuperscript{rd} row and 3\textsuperscript{rd} column into a 1.

QUESTION: If the following system has matrix that reduces to the following, then what is the solution to the system?

\[
\begin{align*}
    x - 3y + z &= 5 \\
    -2x + 7y - 6z &= -9 \\
    x - 2y - 3z &= 6
\end{align*}
\]

\[
\begin{bmatrix}
    | \\
    | \\
    |
\end{bmatrix}
\]

Then, many moves later this can become the following matrix:

\[
\begin{bmatrix}
    | \\
    | \\
    |
\end{bmatrix}
\]

[You Do!]
Gauss-Jordan 3x3 Worksheet

There are at most 9 steps to reduce your augmented (AKA initial) matrix down to a final form. If there is a unique solution, then the final form looks like:

\[
\begin{pmatrix}
1 & 0 & 0 & \ast \\
0 & 1 & 0 & \ast \\
0 & 0 & 1 & \ast \\
\end{pmatrix}
\]

where the * just represent your x, y, and z solutions!

Solve the system

\[
\begin{align*}
x + 2y + 3z &= 3 \\
-2x + y - z &= 3 \\
3x + 4y + 2z &= 6
\end{align*}
\]

The augmented or initial matrix is the following:

\[
\begin{pmatrix}
1 & 2 & 3 & 3 \\
-2 & 1 & -1 & 3 \\
3 & 4 & 2 & 6
\end{pmatrix}
\]

**STEP 1:** Produce a 1 in the 1\(^{st}\) column in the 1\(^{st}\) row. (YAY! IT’S ALREADY THERE!)

**STEP 2:** Do \(2R_1 + R_2 \rightarrow \text{new } R_2\) to get the following:

\[
\begin{pmatrix}
1 & 2 & 3 & 3 \\
0 & 5 & 5 & 9 \\
3 & 4 & 2 & 6
\end{pmatrix}
\]

To the right, show the work.

**STEP 3:** Do \(-3R_1 + R_3 \rightarrow \text{new } R_3\) to get the following:

\[
\begin{pmatrix}
1 & 2 & 3 & 3 \\
0 & 5 & 5 & 9 \\
0 & -2 & -7 & -3
\end{pmatrix}
\]

To the right, show the work.

**STEP 4:** Produce a 1 in the 2\(^{nd}\) column in the 2\(^{nd}\) row by doing \(\frac{1}{5}R_2 \rightarrow \text{new } R_2\)

\[
\begin{pmatrix}
1 & 2 & 3 & 3 \\
0 & 1 & 1 & \frac{9}{5} \\
0 & -2 & -7 & -3
\end{pmatrix}
\]

To the right, show the work.
STEP 5: Do 

\[
\begin{pmatrix}
1 & 2 & 3 & | & 3 \\
0 & 1 & 1 & | & 9/5 \\
0 & 0 & -5 & | & 3/5
\end{pmatrix}
\]

to get the following:

STEP 6: Do 

\[
\begin{pmatrix}
0 & 1 & 1 & | & 9/5 \\
0 & 0 & -5 & | & 3/5
\end{pmatrix}
\]

to get the following:

STEP 7: Produce a 1 in the 3\text{rd} column in the 3\text{rd} row by doing .

STEP 8: Do 

to get the following:

STEP 9: Do 

to get the following:

The answer is \(x=-12/25, y=48/25, \text{ and } z=-3/25.\)
2.3: Arithmetic Operations on Matrices

Definition: A matrix $M$ is any rectangular array of numbers. If $M$ has $m$ rows & $n$ columns, we say $M$ has size $m \times n$.

Example: $M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is a $2 \times 3$ matrix.

$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is a $3 \times 1$ matrix, called a column matrix.

$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ is a $1 \times 3$ matrix, called a row matrix.

Generically, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$

Example: $\begin{bmatrix} 1 & 7 & 4 & 5 \\ 11 & 3 & \frac{1}{2} & 6 \\ -1 & -2 & -3 & -4 \end{bmatrix}$

Then $a_{11} = a_{31} = \text{(You do)}$. $a_{52} = \text{(You do)}$. 

Q: If \( m=n \), then what is the matrix called?

Defn: Two matrices \( A \) & \( B \) are equal if they are the same size & their corresponding entries \( a_{ij} \) & \( b_{ij} \) are equal.

Adding/Subtracting Two Matrices \( A \pm B \)

Only possible if \( A \) & \( B \) are same size

\[
\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 7 \\ 8 & 9 \end{bmatrix} = \begin{bmatrix} 3 & 10 \\ 12 & 14 \end{bmatrix}
\]

Multiplying Two Matrices

Only possible if \( A \) has size \( m \times n \) & \( B \) has size \( n \times p \), then the product has size \( m \times p \).

E.g., \[
\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 & 8 \\ 9 & 10 \end{bmatrix} = \begin{bmatrix} 1 \times 7 + 2 \times 9 + 3 \times 10 & 1 \times 8 + 2 \times 10 + 3 \times 11 \\ 4 \times 7 + 5 \times 9 + 6 \times 10 & 4 \times 8 + 5 \times 10 + 6 \times 11 \end{bmatrix} = \begin{bmatrix} 56 & 64 \\ 139 & 158 \end{bmatrix}
\]

Why is \[
\begin{bmatrix} 3 & 4 & 1 \\ 2 & 1 & 7 \end{bmatrix} \begin{bmatrix} 7 & 8 \\ 9 & 10 \end{bmatrix}
\]
not possible?
Multiplication Examples

1. \[
\begin{bmatrix}
1 & 5 \\
3 & 2
\end{bmatrix}
\cdot
\begin{bmatrix}
1 & 2 \\
1 & 0
\end{bmatrix}
= \begin{bmatrix}
1(1) + 5(1) & 1(2) + 5(0) \\
3(1) + 2(1) & 3(2) + 2(0)
\end{bmatrix}
= \begin{bmatrix}
6 & 2 \\
5 & 6
\end{bmatrix}
\]

2. \[
\begin{bmatrix}
3 & -1 \\
2 & 0 \\
1 & 5
\end{bmatrix}
\cdot
\begin{bmatrix}
5 & 4 \\
-2 & 3
\end{bmatrix}
\]

Question: What size is the product matrix above?

FACT: If A is size m \times n & B is size n \times p
Then AB is size m \times p.

3. \[
\begin{bmatrix}
0 & 1 & 2 \\
-1 & 4 & 1/2 \\
1 & 3 & 0
\end{bmatrix}
\begin{bmatrix}
3 & -1 & 5 \\
0 & 2 & 2 \\
4 & -6 & 0
\end{bmatrix}
\]

[You do!]

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After many eq's of multiplication do the following:

<table>
<thead>
<tr>
<th>Shirt</th>
<th>Ties</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bob</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>Tom</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>Macy's</td>
<td>10</td>
</tr>
<tr>
<td>Sears</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>7</td>
</tr>
<tr>
<td>Walmart</td>
<td>Price per shirt</td>
</tr>
<tr>
<td>Price per tie</td>
<td></td>
</tr>
</tbody>
</table>

Q: interpret the entry $a_{12}$ in the product
A: [you do!]

Representing a System of Equations as a Matrix equation:

\[
\begin{align*}
2x + 4y &= 2 \\
-3x + 7y &= 7 
\end{align*}
\]

Note this can be written as the following two 2x1 matrices:

\[
\begin{bmatrix}
-2x + 4y \\
-3x + 7y 
\end{bmatrix} =
\begin{bmatrix}
2 \\
7 
\end{bmatrix}
\]

Let \( A = \begin{bmatrix} -2 & 4 \\ -3 & 7 \end{bmatrix} \) \( X = \begin{bmatrix} x \\ y \end{bmatrix} \) and \( B = \begin{bmatrix} 2 \\ 7 \end{bmatrix} \)

Then \( AX = B \). 

Q: Why do we do this?
Consider \( 7x = 3 \), how do we solve? Need inverse for 7.
2.4 Inverse of a Matrix

The motivation:

Q: How do you solve $7x = 3$?

A: [You do!]

The essence of the solution is using the INVERSE of 7.

The inverse of 7 is $\frac{1}{7}$.

Why? Because $\frac{1}{7} \times 7$ equals 1.

Q: Can a matrix have an inverse? And if so, then what is the "1" in matrix world?

A: The "1" is called an identity matrix.
FACTS:

1. Only square matrices can have an inverse.

2. But NOT every square matrix has an inverse.

3. A $2 \times 2$ matrix $M$ has an inverse $M^{-1}$ if and only if the Phattie Dee of $M$ is not equal to zero.

**Defn:** Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then the determinant of $M$ is $D = ad - bc$. We call $D$ the term "Phattie Dee"!!

**Defn:** The "1" in Square Matrix Land is called the Identity Matrix.

$I_1 = \begin{bmatrix} 1 \end{bmatrix}$  $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$  $I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$  Etc.
Motivation (Continued):

\( \frac{1}{7} \) is the inverse of 7 since

\[
\frac{1}{7} \times 7 = 1 = 7 \times \frac{1}{7}
\]

Analogue in Matrix Land: Let \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \)

\( A^{-1} \) is the inverse of \( A \) since

\[
A^{-1}A = I_2 = AA^{-1}
\]

Let's Do Math that will help us with the WebWork now!!

Defn: The inverse \( A^{-1} \) of \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) is

\[
A^{-1} = \frac{1}{D} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} \frac{d}{D} & -\frac{b}{D} \\ -\frac{c}{D} & \frac{a}{D} \end{bmatrix}
\]

where \( D = ad - bc \).

[You do!] \( A = \begin{bmatrix} -2 & 4 \\ -3 & 7 \end{bmatrix} \) Find \( A^{-1} \).
Prove $A^{-1}A = I_2$ & $AA^{-1} = I_2$

when $A = \begin{bmatrix} -2 & 4 \\ -3 & 7 \end{bmatrix}
\begin{align*}
\text{The Inverse Matrix Method} \\
\text{Solve} & \quad \begin{cases} -2x + 4y = 2 \\
-3x + 7y = 7 \end{cases} \tag{\star} \\
\text{Let} & \quad A = \begin{bmatrix} -2 & 4 \\ -3 & 7 \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \text{&} \quad \mathbf{B} = \begin{bmatrix} 2 \\ 7 \end{bmatrix}
\end{align*}
\begin{align*}
\text{"coefficient matrix"} & \quad \text{"variables matrix"} \\
\text{"Barney" (i.e., the ANSWER)}
\end{align*}
\begin{align*}
\text{Q: Why is } A\mathbf{X} = \mathbf{B} \text{ the same as system } (\star) \text{?? (see above)}
\end{align*}
\begin{align*}
A: [\text{you do!}]
\end{align*}
Recall: to solve $7x = 3$, you multiply both sides by the inverse of 7.

Analogue: to solve $AX = B$, you multiply both sides by the inverse $A^{-1}$.

To solve \[
\begin{align*}
-2x + 4y &= 2 \\
-3x + 7y &= 7
\end{align*}
\] (\text{x})

Step 1] Find $A$ & $B$

Step 2] We know \( \mathbf{x} = [x, y] \) is the solution. Find it by multiplying both sides of $AX = B$ by $A^{-1}$.

Step 3] The solution is $A^{-1}(AX) = A^{-1}B$

That is, $\mathbf{x} = A^{-1}B$

[you do!] Solve (\text{x}) using this method.
Exercises:

1. Find the inverse of \[
\begin{bmatrix}
2 & 7 \\
1 & -3
\end{bmatrix}
\].

2. Why does \[
\begin{bmatrix}
-6 & -3 \\
4 & 2
\end{bmatrix}
\] NOT have an inverse?

3. Fill in the missing blanks.
\[
\begin{bmatrix}
2 & 3 \\
5 & 7
\end{bmatrix}
\begin{bmatrix}
\square & \square & \square \\
\square & \square & \square
\end{bmatrix}
= \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

4. Solve \[
\begin{cases}
2x + 7y = 13 \\
x - 3y = 26
\end{cases}
\]
using the inverse matrix method.

(Hint: Use exercise 1 at some point)
Hey bub! Why is Phattie Dee called the determinant?

First of all, I ain't your bub. Secondly, it is called determinant because it can determine when an inverse exists & other cool stuff.

**Stuff Phattie Dee determines:**

Given $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

1. $D = 0 \implies A$ has no inverse

2. Given the two lines $\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$

   If $D \neq 0$, then $l_1 \cap l_2$ Unique Solution

   If $D = 0$, then either

   - $l_1 \parallel l_2$ No Solution
   - $l_1$, $l_2$ Infinite Solutions
More Exercises:

1) Without solving the system, prove that the system \( \begin{cases} x + 2y = 3 \\ 2x + 6y = 5 \end{cases} \) has a Unique Solution, i.e., \( \times \) happens.

2) Consider \[ A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix} \] & its inverse \[ A^{-1} = \begin{bmatrix} 5 & -2 & -2 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \]

(i) How would you prove \( A^{-1} \) really is the inverse of \( A \)?

(ii) Use the inverse matrix method to solve the system \[ \begin{cases} x + 2y + 2z = 1 \\ x + 3y + 2z = 0 \\ x + 2y + 3z = 0 \end{cases} \]
Ch. 2 Quiz Review

Gauss Jordan Row Reduction

1. If you did row reduction & got to the following matrix, what is the solution?

\[
\begin{bmatrix}
1 & 0 & 2 & 3 \\
0 & 1 & 3 & 4 \\
0 & 0 & 0 & 5
\end{bmatrix}
\]

2. Same Q as above, but change the matrix to:

\[
\begin{bmatrix}
1 & 0 & 2 & 3 \\
0 & 1 & 3 & 4 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
Does the system have a
1) Unique Solution (U) ?
2) No Solution (N)
or 3) Infinitely Many Solutions (I)

Recall: 1) if \( D = 0 \), then \( \frac{b_1}{b_2} \) or \( \frac{b_2}{b_1} \) happens so there are N or I.
2) if \( D \neq 0 \), then \( \frac{y_1}{x_1} \) happens so U.

A) \[
\begin{cases}
-\frac{1}{2}x + y = \frac{3}{2} \\
-3x + 6y = 10
\end{cases}
\]

B) \[
\begin{cases}
-\frac{1}{2}x + y = \frac{3}{2} \\
-3x + 6y = 9
\end{cases}
\]

C) \[
\begin{cases}
2x + 3y = 9 \\
-x + 6y = 8
\end{cases}
\]
Handout & Group Work for Sect. 2.3 & 2.4

Recall if \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \), then \( A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \)

where \( D = ad - bc \). Note: \( A^{-1} \) does not exist if \( D = 0 \).

Consider the system \((\star)\) \[
\begin{align*}
2x + 2y &= 3 \\
2x + 6y &= 5
\end{align*}
\]

[you do!] Solve using Gauss-Jordan Method

\[ \begin{bmatrix} 2 & 2 & | & 3 \\ 2 & 6 & | & 5 \end{bmatrix} \]

[you do!] Solve using Inverse-Matrix Method

\[ \begin{bmatrix} 2 & 2 \\ 2 & 6 \end{bmatrix}^{-1} \]

[you do!] Why (without much work) do we know the system \((\star)\) has \( \underline{\text{no unique solution}} \)?

\[ \begin{align*}
x + 3y &= 3 \\
2x + 6y &= 5
\end{align*} \]
1. The following two systems have infinitely many solutions.

For (a) write answer as
\[ x = x, \quad y = \text{function of } x \]

For (b) write answer as
\[ x = \text{function of } z, \quad y = \text{function of } z, \quad z = z \]

(a) \[ \begin{aligned} 2x - 4y &= 6 \\ -x + 2y &= -3 \end{aligned} \]

(b) \[ \begin{aligned} x + 5y + 3z &= 9 \\ 2x + 9y + 7z &= 5 \end{aligned} \]
Write both systems from 1 in the form $A \cdot X = B$

where $A$ is the coefficient matrix, $X$ is the column matrix corresponding to the variables, and $B$ is the column matrix corresponding to the right side of the equal symbols in the systems above. For (a), solve the system using the inverse-matrix method.
3. Find the product \[
\begin{bmatrix}
4 & 1 & 0 \\
-2 & 0 & 3 \\
1 & 5 & -1
\end{bmatrix}
\begin{bmatrix}
5 \\
1 \\
2
\end{bmatrix}
\]

4. Find the product \[
\begin{bmatrix}
-1 & 4 & 1/2 \\
1 & 3 & 0
\end{bmatrix}
\begin{bmatrix}
3 & -1 \\
0 & 2 \\
4 & -6
\end{bmatrix}
\]

5. What is the size of a $7 \times 3$ matrix times a $3 \times 4$ matrix?
3.1 A Linear Programming Problem

Let us begin with a detailed discussion of a typical problem that can be solved by linear programming.

Furniture Manufacturing Problem

A furniture manufacturer makes two types of furniture—chairs and sofas. For simplicity, divide the production process into three distinct operations—carpentry, finishing, and upholstery. The amount of labor required for each operation varies. The manufacturer of a chair requires 6 hours of carpentry, 2 hours of finishing, and 1 hour of upholstery. The amount of upholstery required for a sofa is 3 hours of carpentry, 1 hour of finishing, and 2 hours of upholstery. The profit per chair is $80, and the profit per sofa is $70. How many chairs and how many sofas should be produced each day to maximize the profit?
Is This Page Intentionally Left Blank for Your Doodling Delight?

Nope. It is placed here to ensure that the new section on the next page begins with an odd-numbered page.
Recall the furniture problem. We had the following grid:

<table>
<thead>
<tr>
<th></th>
<th>Chairs</th>
<th>Sofas</th>
<th>Available Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Carpentry</td>
<td>6 hrs/chair</td>
<td>3 hrs/sofa</td>
<td>96 hrs</td>
</tr>
<tr>
<td>Finishing</td>
<td>1 hr/chair</td>
<td>1 hr/sofa</td>
<td>18 hrs</td>
</tr>
<tr>
<td>Upholstery</td>
<td>2 hrs/chair</td>
<td>6 hrs/sofa</td>
<td>72 hrs</td>
</tr>
<tr>
<td>Profit</td>
<td>$80/chair</td>
<td>$70/sofa</td>
<td></td>
</tr>
</tbody>
</table>

Let \( x = \) number of chairs made and \( y = \) number of sofas made per day.

**QUESTION**: How many chairs and how many sofas should be produced each day to maximize the profit?

Translating to math, we get what 5 inequalities? And what is the objective function \( P(x,y) \)?
Write the system in standard form and graph the lines. Below we found the intercepts. You find the intersections between the lines.
The Fundamental Theorem of Linear Programming: The maximum (or minimum) value of the objective function is achieved at one of the vertices of the feasible set.

QUESTION: What is the solution to the furniture problem?

The 4-step method to solving Optimization Problems:

Step 1  Translate the problem into mathematical language.
   A. Organize the data.
   B. Identify the unknown quantities, and define corresponding variables.
   C. Translate the restrictions into linear inequalities.
   D. Form the objective function.

Step 2  Graph the feasible set.
   A. Put the inequalities in standard form.
   B. Graph the straight line corresponding to each inequality.
   C. Determine the side of the line belonging to the graph of each inequality. Cross out the other side. The remaining region is the feasible set.

Step 3  Determine the vertices of the feasible set.

Step 4  Evaluate the objective function at each vertex. Determine the optimal point.

NOTE: Only use Desmos to confirm your results. Do not simply use Desmos to “cheat” and get a quick answer. On quizzes/exams, you MUST show your work.
GROUP WORK TIME:

7. Exam Strategy A student is taking an exam consisting of 10 essay questions and 50 short-answer questions. He has 90 minutes to take the exam and knows he cannot possibly answer every question. The essay questions are worth 20 points each and the short-answer questions are worth 5 points each. An essay question takes 10 minutes to answer and a short-answer question takes 2 minutes. The student must do at least 3 essay questions and at least 10 short-answer questions.

(a) Fill in the following chart.

<table>
<thead>
<tr>
<th></th>
<th>Essay questions</th>
<th>Short-answer questions</th>
<th>Available</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time to answer</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Quantity</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Required</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Worth</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(b) Let $x$ be the number of essay questions to be answered and $y$ the number of short-answer questions to be answered. Refer to the chart and give the inequality that $x$ and $y$ must satisfy due to the amount of time available.

(c) Give the inequalities that $x$ and $y$ must satisfy because of the numbers of each type of question and also because of the minimum number of each type of question that must be answered.

(d) Give an expression for the score obtained from answering $x$ essay questions and $y$ short-answer questions.

(e) Graph the feasible set for the exam strategy problem.

Find all points of intersection
The 4-Step Method for Solving Optimization Problems (pg. 117 of textbook)

Chapter 3

Step 1  Translate the problem into mathematical language.
  A.  Organize the data.
  B.  Identify the unknown quantities, and define corresponding variables.
  C.  Translate the restrictions into linear inequalities.
  D.  Form the objective function.

Step 2  Graph the feasible set.
  A.  Put the inequalities in standard form.
  B.  Graph the straight line corresponding to each inequality.
  C.  Determine the side of the line belonging to the graph of each inequality. Cross out the other side. The remaining region is the feasible set.

Step 3  Determine the vertices of the feasible set.

Step 4  Evaluate the objective function at each vertex. Determine the optimal point.
Chapter 3 – Problems similar to the WeBWorK

1. A manufacturer of downhill and cross-country skis reports that manufacturing time is 3 hours and 5 hours, respectively, per ski and that finishing time is 4 hours for each downhill and 3 hours for each cross-country ski. There are only 35 hours per week available for the manufacturing process and 32 hours for the finishing process. The average profit is $55 for downhill ski and $62 for cross-country ski.

The manufacturer wants to know how many of each type of ski should be made to maximize the weekly profit.

Let \( x \) = the number of downhill skis, and let \( y \) = the number of cross-country skis. Solve this optimization problem.

Answer: Max P is $523 at \( x=5 \) and \( y=4 \).
2. A diet is to contain at least 1031 units of carbohydrates, 1652 units of proteins, and 1914 calories. Two foods are available: $F_1$ which costs $0.02 per unit and $F_2$, which costs $0.06 per unit. A unit of food $F_1$ contains 1 units of carbohydrates, 2 units of proteins and 4 calories. A unit of food $F_2$ contains 9 units of carbohydrates, 8 units of proteins and 6 calories.

**Find the minimum cost for a diet that consists of a mixture of these two foods and also meets the minimal nutrition requirements.**

**HINT for Math 104 Students:** Follow the instructions in the brown box on pgs.117. For example, for Step 1(B), let $x = F_1$ = the number of units of Food 1, and let $y = F_2$ = the number of units of Food 2. For Step 1(C), translate the restrictions into linear inequalities (in addition to the obvious inequalities $x \geq 0$ and $y \geq 0$, you should have 3 other linear inequalities in this problem. For example, the protein constraint says that $2x + 8y \geq 1652$. **What are the other two inequalities?**). For Step 1(D), form the objective function (in this problem, it would be the cost function $C = 0.02x + 0.06y$). Then perform Steps 2, 3, and 4. See Example 1 on pgs.118-119 to see a similar problem. Please see me or email me if you have any trouble with these types of problems.

Answer: Max $C$ is $13.74$ when $F_1=270$ and $F_2=139$. 

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The feasible set for Problem 2

Note: This is called an unbounded feasible set. Why do you think?
Group Work

3. Blink Appliances plans to order microwaves and stoves. Each microwave requires 1.5 hours to unpack and set up in the storeroom, and each stove requires 3 hours. The storeroom space is limited to 45 items. The budget of the store allows only 84 hours of employee time for unpacking and setup. Microwaves yield a profit of $95 each, and stoves yield a profit of $165 each. How many of each should the store order to maximize profit?

**Math 104 Students:** Like the previous problem, follow the steps in the brown box on pgs.117. It might be helpful to label the variables as follows $m =$ number of microwaves, and $s =$ number of stoves. Then we have the objective profit function $P = 95m + 165s$. The obvious constraints are the linear inequalities $m \geq 0$ and $s \geq 0$.

I leave it to you to write the two other linear inequalities and solve the problem.

Answer: Max P is $5045$ at $m=34$ and $s=11$. 

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Is This Page Intentionally Left Blank for Your Doodling Delight? No. It is placed here to ensure that the new section on the next page begins with an odd-numbered page.
Ch. 3 Worksheet

1. Minimize the objective function \( f(x, y) = 3x + 4y \) subject to the constraints:
   \[
   \begin{align*}
   2x + y & \geq 10 \\
   x + 2y & \geq 14 \\
   x & \geq 0, \ y & \geq 0
   \end{align*}
   \]

   
   \[\text{Answer: } f \text{ has min value of 30 at } (2, 6)\]

   [For full credit on Exam I, ALL points of intersection AND x & y-intercepts must be found]
(2) Maximize the objective function \( O(x, y) = 2x + 5y \)
subject to the constraints \[
\begin{align*}
  x + 2y & \leq 20 \\
  3x + 2y & \geq 24 \\
  x & \leq 6
\end{align*}
\]

Answer: \( O \) has a max value of 49 at \((2, 9)\)

[Again on Exam I, all intercepts & intersection points must be labeled and all work shown.]
3. Find the three inequalities that have the following feasible set. \[ \text{NOTE: We shade the feasible set in this instance, just like the WEBWORK does.} \]

When possible, write your answers in general form:
\[
ax + by \leq c
\]
\[
or \quad ax + by \geq c.
\]
12. Political Campaign—Resource Allocation A local politician has budgeted at most $80,000 for her media campaign. She plans to distribute these funds between TV ads and radio ads. Each one-minute TV ad is expected to be seen by 20,000 viewers, and each one-minute radio ad is expected to be heard by 4000 listeners. Each minute of TV time costs $8000, and each minute of radio time costs $2000. She has been advised to use at most 90% of her media campaign budget on television ads.

(a) Fill in the following chart. *(Note: Fill in only the first entry of the last column.)*

<table>
<thead>
<tr>
<th></th>
<th>One-minute TV ads</th>
<th>One-minute radio ads</th>
<th>Money available</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cost</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Audience reached</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Let \( x \) = # of minutes of TV ads
Let \( y \) = # of minutes of radio ads.

Solve the problem (i.e., find maximum allocations of TV and radio ads funds):

Answer: 9 min of TV & 4 min of Radio

i.e., Allocate $72,000 for TV & $8000 for radio
5.1 The Joy of Sets

Defn: A set is any collection of objects. These objects are called elements of the set.

Examples:

Warning! Order does not matter and duplicate elements are ignored.

\[ \{ y, z, x \} = \{ x, y, z, z \} \quad \text{True or False} \]

\[ \{ 1, \text{one}, 0 \} = \{ \text{one}, 0 \} \quad \text{True or False} \]

Let \( A = \{ x, y, z \} \), \( B = \{ 2, 4, 6, 8, 16, 32 \} \), and \( C = \{ \text{the odd integers} \} \).

Q: Which of the sets above are finite and which are infinite.
Compact ways to write sets

Recall $B = \{2, 4, 6, 8, 16, 32\}$.

We could also write $B = \{2^k \mid k \text{ is an integer and } 1 \leq k \leq 5\}$

This is called set-theoretic notation.

Recall $C = \{\text{the odd integers}\}$

$= \{\ldots, -7, -5, -3, -1, 1, 3, 5, 7, \ldots\}$

[You do!] Write $C$ in set-theoretic notation.

Prepare to learn an ARSENAL of set theory symbols: $\in$, $\cup$, $\cap$, $\subseteq$, $\emptyset$, etc.

Q1 Is 16 an element of $C$?
How to express this in set theory lingo?

Q2 Is $B$ contained in $C$?
How to express this in set theory lingo?
Q3: Do B and C contain common elements? How to express this in set theory lingo?

Q4: What is not in C? How to express this in set theory lingo?

The Symbol \( \in \) ("an element of")

Let \( C = \{ \text{odd integers} \} \)

\( 3 \in C \) is said as "3 is an element of C"

Q: Why is 30 \( \notin C \)?

The Symbol \( \cup \) ("union") and \( \cap \) ("intersection")

Let \( A \) and \( B \) be two sets

Defn: \( A \cup B \) (said as "A union B") is the set of all elements that belong to either \( A \) or \( B \) (or possibly both).
Def: $\text{A} \cap \text{B}$ (said as "A intersection B") is the set of all elements that belong to both $\text{A}$ and $\text{B}$.

**Memory Tip:**

- $\cup \leftrightarrow \text{OR}$
- $\cap \leftrightarrow \text{AND}$

Exercise: (You do!)

$A = \{1, 2, 3, 4\}$ & $B = \{1, 3, 5, 7, 11\}$

$A \cup B =$

$A \cap B =$

**The Symbol $\subseteq$ ("a subset of")**

Def: $B \subseteq A$ (said as "B is a subset of A") means every element of B is also an element of A.

e.g. $B = \{1, 2\}$ & $A = \{3, 5, 1, 7, 2\}$

$B \subseteq A$ True or False (You do!)
The Symbols $\emptyset$ ("the empty set") & $U$ ("the universal set")

**Def.** The empty set is the set with no elements. It is denoted $\emptyset$ or $\varnothing$.

**Def.** The universal set (or "the universe") is the superset that a set's elements come from. It is denoted $U$.

**Note:** $U$ is understood by the situation.

The Complement of a Set

**Def.** $A'$ (pronounced "$A$-complement") is the set of all elements in the universe $U$ that do not belong to $A$.

**Exercise:** $E = \{\text{even integers}\}$

Describe $U$ and $E'$. (You do!)
Worked Examples: (you do!)

Let $U = \{a, b, c, d, e, f, g\}$
$S = \{a, b, c\}$ and $T = \{a, c, d\}$

1. $S' =$

2. $T' =$

3. $(S \cap T)' =$

4. $S' \cap T' =$

5. $S' \cup T' =$

6. All subsets of $S$ are?
5.1 Joy of Sets (continued)

Popular Numerical Sets
1. True/False (and give a reason why)

1) $5 \in \{3, 5, 7\}$
2) $\emptyset \in \{3, 5, 7\}$
3) $5 \subseteq \{3, 5, 7\}$
4) $\{5\} \subseteq \{3, 5, 7\}$
5) $0 \in \emptyset$
6) $\emptyset \subseteq \{a, b, c\}$

2. Let $U = \{1, 2, 3, 4, 5\}$, $R = \{1, 3, 5\}$, $S = \{3, 4, 5\}$, and $T = \{2, 4\}$. Find the following sets (the 1st is done for you).

a) $S' \cap T = \{2\}$

b) $R \cap S \cap T' = \emptyset$

c) $R \cap S \cap T'$

d) $R \cap S' \cap T$

e) $R' \cap T$

f) $R \cup S = \{1, 2, 3, 4, 5\}$

g) $(S \cap T)' = \{2, 4\}$

h) $R' \cup U = \{1, 2, 4\}$

i) $S' \cup T'$

j) $R \cap R' = \emptyset$

3. How many elements in $\emptyset$?

How many elements in $\{\emptyset\}$?
How many elements in $\{\emptyset, \emptyset, \emptyset\}$?
5.2: A Fundamental Principle of Counting

**Def.** The size (or cardinality) of a set $S$ is the number of elements in $S$. It is denoted $n(S)$, or $N(S)$, or $|S|$. 

**Exercise:** You do! 

1. $S = \{a, 1, dog\} \implies n(S) =$

2. $S = \{\text{even integers}\} = 2\mathbb{Z} \implies n(S) =$

3. $S = \emptyset \implies n(S) =$

4. $S = \{\emptyset\} \implies n(S) =$

5. $S = \{\emptyset, \emptyset \emptyset, \emptyset, \emptyset \emptyset \emptyset \emptyset\} \implies n(S) =$

6. $S = \{7, 7, 7, 7\} \implies n(S) =$

7. $S = \{\text{even integers}\} \cap \{\text{odd integers}\} \implies n(S) =$
The Inclusion-Exclusion Principle

Let $S$ and $T$ be sets.

Then $n(S \cup T) = n(S) + n(T) - n(S \cap T)$

**Exercises [you do!]

1. Suppose $S = \{a, b, c, d\}$ and $T = \{c, d, e\}$.

Then $n(S \cup T) =$

2. Suppose 1000 students are enrolled in a Math or English course. Suppose 400 are taking both Math and English. Suppose 600 are taking English. How many are taking a Math course?

**Solution:**
De Morgan's Laws for Sets:

Let $S$ and $T$ be sets. Then [you do!]

$$(S \cup T)' =$$

$$(S \cap T)' =$$

Exercise: [You do!]

200 UWEC students were surveyed on whether or not they are planning to go to the upcoming basketball or football game. 58 students said they would miss at least one of the two games (note: this includes those planning to miss both games).

**QUESTION:** How many students plan to attend both games? (Use De Morgan's Laws)
Venn Diagrams or "bubbles"

Venn diagrams are graphical representations of sets.

Example: Characteristics of Mammals & Fish

- **Mammals**
  - Warm-blooded
  - Have hair or fur
  - Have live births
  - Breathe with lungs

- **Vertebrates (have a backbone)**

- **Fish**
  - Cold-blooded
  - Scaly skin
  - Hatch from eggs
  - Breathe with gills

**Exercises**:

1. Shade S

2. Shade T
3. Shade SNT

4. Shade S\text{UT}

5. Shade T'

6. Shade S\text{NT}'

7. Shade S'\text{UT}'}
THINGS LIKED BY
Nerds, Goths, and Jocks

Nerds
- Programming
- RPGS
- SciFi
- Shirts That Are Too Big
- Cosplay

Goths
- Makeup
- Satan
- JNCOs

Jocks
- Sports
- Dave Matthews
- Fighting
- The Mall
- Shitty Metal Bands

Criticizing Other People
A worksheet to practice Ch.5 (sections 1, 2 and 3 stuff)

Part 1: 
Shade the region of the Venn diagram indicated by the following sets.

(i) Shade: \((A' \cup B) \cap C\)

(ii) Shade: \((A \cap B)' \cup C\)

(iii) Shade: \(A \cup (C' \cup B)\)

(iv) Shade: \((A \cap B') \cap C\)
**Part II:** Write down the elements in the following sets.

Let \( U = \{0,1,2,3,4,5,6,7,8,9,10\} \); \( A = \{0,1,2,3,5,8\} \); \( B=\{0,2,4,6\} \); \( C = \{1,3,5,7\} \)

i) \( A \cup B = \) 

ii) \( B' = \) 

iii) \( A \cap B' = \) 

iv) \( A' \cup C = \) 

v) \( (A \cup B)' = \) 

vi) \( (A \cup C) \cap B = \) 

**Part III:** Refer to the diagram to answer the questions below.

![Venn Diagram]

What set notation would you use to represent the following regions? (I.e. if those regions were shaded, what set notation would you use to identify the shaded region). Hint – make a separate sketch for each.

Example: Region 3 could be written as \( A \cap B \)

i) Region 1,2 and 4 (i.e. imagine all region 1, 2 and 4 were shaded):

ii) Region 2 only:

iii) Region 1 only:

iv) Region 1 and 4:
**Part IV:** Refer to the diagram to answer the questions below.

What set notation would you use to represent the following regions? (I.e. if those regions were shaded, what set notation would you use to identify the shaded region).

i) Region 7 only:

ii) Region 1 only:

iii) Regions 1 and 4:

iii) Region 2 only:

iv) Regions 4, 5, 6, 7 & 8

**Part V:** An inclusion-exclusion principle problem

Suppose A and B are sets and that the following holds:
- \( n(A \cap B) = 6 \)
- \( n(A) = 14 \)
- \( n(A \cup B) = 40 \)

What is the value of \( n(B) \) (use the Inclusion-Exclusion formula)?

What is the value of \( n(B) \) (use a Venn diagram)?
(3 points) Given any set $X$, the complement of $X$ is denoted $X'$ and is defined on pg.192 of the book.

Let $U = \text{Universal set} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{1, 3, 4, 5, 6, 8\}$, and $B = \{0, 3, 6, 8\}$.

List the elements of the following sets in the increasing order:

$A' = \{\hspace{1cm}, \hspace{1cm}, \hspace{1cm}, \hspace{1cm} \}$

$(A \cup B)' = \{\hspace{1cm}, \hspace{1cm}, \hspace{1cm} \}$

$(A \cup B')' = \{\hspace{1cm} \}$

$A \cap B' = \{\hspace{1cm}, \hspace{1cm}, \hspace{1cm} \}$

**TO DO:** Draw the corresponding Venn diagrams below.
5.1 Sets: Problem 7

(3 points) Let $U =$ Universal set $= \{a, b, c, d, e, 1, 2, 3\}$, $A = \{2, d, b, e, 3\}$, and $B = \{1, d, a\}$.
List the elements of the following sets. If there is more than one element write them separated by commas.

$A' = \{\}$

$(A \cup B)' = \{\}$

$A' \cap B' = \{\}$

**TO DO:** Draw the corresponding Venn diagrams below.

5.1 Sets: Problem 8

(3 points) Let

$U =$ Universal Set $= \{\text{All college students}\}$

$M =$ \{\text{All male students}\}$

$S =$ \{\text{All students who smoke}\}$

$F =$ \{\text{All Freshmen}\}$

Give a description of "all female freshmen students" in terms of a set

- A. $M \cap S$
- B. $F \cap S \cap M$
- C. $M' \cap S'$
- D. $M' \cap F$

**TO DO:** Draw the corresponding Venn diagram below.
The Genesis-Exodus-Leviticus Set Theory Problem

**Reminder for Math 104 Students:** The video recommended in Problem 1 can help you do this and the next problem. Get to the video by [CLICKING HERE](#) in a new window or tab. Minute marker 2:17 is where content related to this problem begins.

167 UWEC students were surveyed to determine if they regularly read the Old Testament Books: Genesis, Exodus, or Leviticus (the first three Books of the five Books of Moses, i.e, the written Torah or Hebrew Bible).

Here were the findings. 72 read Genesis, 71 read Exodus, 40 read Leviticus, 39 read exactly two of the three Books, 25 read Genesis and Exodus, 26 read Exodus and Leviticus, and 7 read all three Books.

How many read none of the three Books?

How many read exactly one of the three Books?

How many did not read Leviticus?

If you are curious, [HERE](#) is a list of 76 things banned in the Book of Leviticus.

Meow. Please don’t eat me. I swear that I don’t taste like chicken.
5.4 Multiplication Principle

Q: How many distinct paths from A → B → C?

Schneider

A: Centennial  B: Schoefield  C: Library

Hibbard Hall

Q: How many ways to line up 3 students in a row?

Q: (Same Q above, but with 4 students). [Divide problem into 4 cases dependent on where the new student stands.]
Observation: We added the $6 + 6 + 6 + 6$ because each case was a separate counting job or problem in itself. [This is the **Addition Principle**.]

Reconsider the 4-students-in-a-row Problem

- **Step 1:** How many choices for this person?
- **Step 2:** How many choices for this person?
- **Step 3:** How many choices for this person?
- **Step 4:** How many choices for this person?

We are computing the ways to do one job/task but have divided this job into 4 steps ("consecutive minitasks"). At the end of the 4th & final step, we are done with the ONE job. That is why we multiply. [This is the **Multiplication Principle**.]

**Total Ways to line up 4 students** =
**Generalized Multiplication Principle:**

Suppose a job (or task) can be composed of \( t \) consecutive steps (or “minitasks”). And

- Minitask #1 can be done in \( m_1 \) ways.
- Minitask #2 can be done in \( m_2 \) ways.
- Minitask #3 can be done in \( m_3 \) ways.
- Etc...  
- Minitask \( t \) can be done in \( m_t \) ways.

Then the TOTAL WAYS to do the job =

**Some Exercises:**

1. How many ways to form a 6-letter word using the letters A, B, C, D, E, F, G, and H if repetition is allowed?

2. How many ways to form a 6-letter word using the letters A, B, C, D, E, F, G, and H if repetition is NOT allowed?

3. How many ways to form a 6-letter word using the letters A, B, C, D, E, F, G, and H if repetition is allowed [AND] the word must start with one or two A’s?

KFCat says separate into two cases and use ADDITION PRINCIPLE!
For the questions 4 thru 8 consider the set \{1, 2, 3, ..., 9\}

4. How many 9-digit numbers are there (if repetition is allowed)?

5. How many 9-digit numbers are there (if repetition is NOT allowed)?

6. How many 4-digit numbers are there (if repetition is allowed)?

7. How many 4-digit numbers are there (if repetition is NOT allowed)?

8. How many 4-digit numbers between 8000 and 8999 (if repetition is allowed)?

9. How many 4-digit numbers are even and larger than 8000 (if repetition is NOT allowed)? **Let the set \{0,1,2,3, ..., 9\} include zero this time.**
5.5 Permutations & Combinations

Def: A permutation is an arrangement of objects when order matters.

E.g. There are 6 permutations of the letters a, b, c:
    abc  bac  cab
    acb  bca  cba

Def: A combination is a collection of objects where order does not matter.

E.g. The 6 winning lottery numbers \{7, 25, 5, 16, 4, 50\}

Three Important Formulas

1. n-factorial is \( n! \) & \( n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1 \)

   E.g. \( 0! = 1 \) \( 1! = 1 \) \( 2! = 2 \times 1 \) \( 3! = 3 \times 2 \times 1 \) \( 4! = 4 \times 3 \times 2 \times 1 \)

2. \( P(n, r) = \) the # of permutations of \( n \) objects taken \( r \) at a time

   \[ = \frac{n!}{(n-r)!} \]

3. \( C(n, r) = \) the # of combinations of \( n \) objects taken \( r \) at a time

   \[ = \frac{n!}{r! \cdot (n-r)!} \]
A connection between $P(n,r)$ & $C(n,r)$

Q: What does $P(7,4)$ mean?

A: [Two Minitasks view]

The Job: "to count the number of permutations of size four in a set with 7 elements."

That is, how many ways can I arrange all the possible subsets of 4 elements in a set with 7 elements?
Exercises: [you do!]

1. \[ P(1000, 1) = ?? \]

What a 1000! is a very calculator-crushingly large number!

You do not need calculators for this one, dear.

2. Calculate \( P(8, 3) \).

3. Calculate \( C(8, 3) \).

4. Calculate \( C(8, 3) \times 3! \). Why is it same as \( P(8, 3) \)?
5) In general, \( P(n,r) = C(n,r) \times r! \). Why?

6) How many different selections of two different killer bunnies can be made from nine distinct killer bunnies?

7) How many ways can 3 of a total of 5 siblings line up for a group photo?
1. Eight Math 104 students enter a math competition. Assuming no ties, how many ranking outcomes of the 1\textsuperscript{st}, 2\textsuperscript{nd}, and 3\textsuperscript{rd} place can happen? \textbf{DOES ORDER MATTER?}

2. In a class of 8 students, how many ways can we select a group of 3 students? \textbf{DOES ORDER MATTER?}

3. In a class of 8 students, how many ways can we select a Chair, Vice Chair, and Secretary? \textbf{DOES ORDER MATTER?}
4. If you have 6 books, in how many ways can you select 4 books and arrange them on the shelf? (Do this in two different ways!)

5. An experiment consists of tossing a coin 9 times and observing the sequence of heads and tails. (NOTE: the choice of the word “sequence” denotes that order matters).

   a) How many different outcomes are possible? (HINT: Start by drawing some!)

   b) How many different outcomes have exactly two tails? (HINT: Start by drawing some!)

   c) How many different outcomes have exactly two tails, and the tails are adjacent to each other? (HINT: Start by drawing some!)
d) How many outcomes have at least 7 heads?
*(NOTE: aBa will talk about addition principle here since we will split this problem up into 3 disjoint cases.)*

e) How many outcomes have at most 6 heads?
*(NOTE: aBa will talk will explain how part (a) and (d) can simplify this problem.)*

First I will show you why chopping this into 7 different cases will work, but is not the best method to do this problem.

6. How many 6 digit numbers are there that have the digits strictly decreasing when read left to right? (e.g., 854210 or 987654 or 654321, etc.)
*(HINT: Consider the 10 digit number 9876543210, and ask yourself how many ways to remove 4 digits?)*
Is This Page Intentionally Left Blank for Your Doodling Delight?

Nope. It is placed here to ensure that the new section on the next page begins with an odd-numbered page.
[YOU DO!]

Recall the some important "formulas" (i.e., counting concepts):

1. The number of ways to arrange $n$ distinct objects is:

   [What is the above thing called?]

2. The number of ways to select $r$ objects from $n$ distinct objects and then arrange these $r$ objects is:

   [What is the above thing called?]

3. The number of ways to select $r$ objects from $n$ distinct objects.

   [What is the above thing called?]

4. The number of ways to distribute $r$ distinct objects into $n$ different slots?
The Chipotle Menu Counting Problem

Here is a menu for Chipotle. You choose your style of meal, meat, salsa, and if you want extras.

5. How many different meals can we order if don’t want extras and can only choose one thing from each category?

6. My g-friend B tells me that I have an old menu, and Safritas (some organic tofu thing) is an available “meat option”. Answer the question above considering this new menu option.
7. How many different burritos can be ordered if the Chipotle manager says that you can only choose at most two meat options on your one burrito, only one salsa, and NO extras (NOTE: we count *Safritas* as a meat option).

8. How many paths are there from A to B?
9. What if you are a parent walking with your child, and there is a nasty Drug Dealer at location DD. How many paths avoid the drug dealer?

HELPFUL HINT:
Let $E = \{\text{set of paths from } A \rightarrow DD \rightarrow B\}$. Let $U = \{\text{set of all paths from } A \rightarrow B\}$. In Question 8, you calculated $n(U)$. Then the answer you seek is $n(E')$ and that equals $n(U) - n(E)$. (Why?)
10. How 5-card poker hands are there?

11. How many 5-card poker hands contain exactly 1 heart?

12. How many 5-card poker hands are full houses?
13. An urn contains 25 numbered balls, of which 15 are red and 10 are white. A sample of 5 balls is to be selected.

(a) How many different samples are possible?

(b) How many samples contain all red balls?

(c) How many samples contain 3 red balls and 2 white balls?

(d) How many samples contain at least 4 red balls?
Group Work Sheet on Counting Problems

Math Bldg (MB)

East (E)

South (S)

Teddy Bear Shop (TB)

For ALL questions below, suppose you can only walk south (S) or east (E).

1. How many paths are there from MB to TB?

2. Suppose a drug dealer at location DD desires to visit his gang banger friend at location GB. How many paths are there?

3. How many paths from MB to TB avoid both the DD & GB locations?
4. How many different 6 card poker hands are there?

5. How many six card poker hands have a three of a kind.

6. How many 4-of-a-kind hands in a 5-card poker hand?
Binomial Coefficients

Recall \( \binom{n}{r} \) "n choose r"

Q: What is the formula for \( \binom{n}{r} \)?

A: [you do!]

Who is Blaise Pascal?

"The heart has its reasons which reason knows nothing of... We know the truth not only by the reason, but by the heart."

-Blaise Pascal (1623—1662)
Pascal's Triangle:

1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

Q: How is this table constructed?
A: [You do!]

\[
\begin{array}{cccccc}
(0) & (1) & (2) & (3) & (4) & (5) \\
(0) & (1) & (2) & (3) & (4) & (5) \\
(0) & (1) & (2) & (3) & (4) & (5) \\
(0) & (1) & (2) & (3) & (4) & (5) \\
(0) & (1) & (2) & (3) & (4) & (5) \\
(0) & (1) & (2) & (3) & (4) & (5) \\
\end{array}
\]

In general, \( \binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r} \).

We do: Show \( \binom{5}{3} = \binom{4}{2} + \binom{4}{3} \) via example with 5 people in class and distinguish one of them.
Ah, the Power of Pascal’s Triangle. Tiz a lovely thing that you can COUNT on!

Pascal’s Wager: the argument that it is in one’s own best interest to behave as if God exists, since the possibility of eternal punishment in hell outweighs any advantage of believing otherwise.

- Blaise Pascal (1623-1662), France
- Made advances in geometry at 13
- Discovered properties of fluids and atmospheric pressure
- Invented the first accurate calculator
- At 27, abandoned math and science to study religion; Pascal’s Wager
Consider the expansion of the binomial \((a + b)^n\).

\[
(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k
\]

YIKES! That's the scariest formula ever!

[You do!] Expand \((a + b)^2\) and \((a + b)^3\)

Here are the first few expansions for \(n = 0, 1, 2, \ldots, 6\).

\[
\begin{align*}
(a + b)^0 &= 1 \\
(a + b)^1 &= a + 1b \\
(a + b)^2 &= a^2 + 2ab + b^2 \\
(a + b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \\
(a + b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \\
(a + b)^5 &= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5
\end{align*}
\]

**QUESTION:** Why is \(\binom{n}{r}\) called a binomial coefficient?

Row 0:

\[
\binom{0}{0}
\]

Row 1:

\[
\binom{1}{0} \quad \binom{1}{1}
\]

Row 2:

\[
\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}
\]

Row 3:

\[
\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}
\]

Row 4:

\[
\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}
\]

Row 5:

\[
\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}
\]

Row 6:

\[
\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6}
\]

Row 7:

\[
\binom{7}{0} \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad \binom{7}{7}
\]

Row 8:

\[
\binom{8}{0} \quad \binom{8}{1} \quad \binom{8}{2} \quad \binom{8}{3} \quad \binom{8}{4} \quad \binom{8}{5} \quad \binom{8}{6} \quad \binom{8}{7} \quad \binom{8}{8}
\]

Row 9:

\[
\binom{9}{0} \quad \binom{9}{1} \quad \binom{9}{2} \quad \binom{9}{3} \quad \binom{9}{4} \quad \binom{9}{5} \quad \binom{9}{6} \quad \binom{9}{7} \quad \binom{9}{8}
\]

Row 10:

\[
\binom{10}{0} \quad \binom{10}{1} \quad \binom{10}{2} \quad \binom{10}{3} \quad \binom{10}{4} \quad \binom{10}{5} \quad \binom{10}{6} \quad \binom{10}{7} \quad \binom{10}{8} \quad \binom{10}{9} \quad \binom{10}{10}
\]
In general, we have
\[(a+b)^n = \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{n} b^n\]

The above is the binomial expansion of \((a+b)^n\).

Notice the following:
1. The powers of "a" decrease from \(n\) to 0.
2. The powers of "b" increase from 0 to \(n\).

Example \((n=4)\)
\[(a+b)^4 = \binom{4}{0} a^4 + \binom{4}{1} a^3 b + \binom{4}{2} a^2 b^2 + \binom{4}{3} a b^3 + \binom{4}{4} b^4\]

Expand \((-2x+3y)^4\) [You do!]
Another view of \((a+b)^n\)

\[(a+b)^n = (\binom{n}{0})a^n + (\binom{n}{1})a^{n-1}b + \ldots + (\binom{n}{n-k})a^{n-k}b^k + \ldots + (\binom{n}{n})b^n\]

Q: What is the coefficient of \(x^{13}y^4\) in \((3x-y)^7\) ?

Q: What is the coefficient of \(x^{13}y^3\) in \((3x-y)^7\) ?
Row Sums in Pascal's Triangle

1 1 \rightarrow \text{sum} = 2
1 2 1 \rightarrow \text{Sum} =
1 3 3 1 \rightarrow \text{Sum} =
1 4 6 4 1 \rightarrow \text{Sum} =
1 5 10 10 5 1 \rightarrow \text{Sum} =

Fact: The sum of the numbers in the $n^{th}$ row of Pascal's triangle is ___.

e.g., \((3)^0 + (3)^1 + (3)^2 + (3)^3 = 2^3\)

Question: How many different pizza pies can be ordered if a pizza parlor offers 5 possible toppings?
Star of David Theorem

\[
\begin{array}{cccccc}
1 & & & & & \\
1 & 1 & & & & \\
1 & 2 & 1 & & & \\
1 & 3 & 3 & 1 & & \\
1 & 4 & 6 & 4 & 1 & \\
1 & 5 & 10 & 10 & 5 & 1 \\
1 & 6 & 15 & 20 & 15 & 6 & 1 \\
1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 \\
1 & 8 & 28 & 56 & 70 & 56 & 28 & 8 & 1 \\
1 & 9 & 36 & 84 & 126 & 126 & 84 & 36 & 9 & 1
\end{array}
\]

**QUESTION:** Multiply the numbers in each vertex of the left-pointing triangle. Then, do the same for the right-pointing triangle. What is happening here? Can you come up with a general formula for this phenomena, if for example, the center of the star is the value “\( n \ choose \ r \)”. 
5.8 Multinomial Coefficients

The BANANA problem (essentially)
Pascal’s Tetrahedron

\[(A + B + C)^0 = 1\]
\[(A + B + C)^1 = A + B + C\]

\[(A + B + C)^2 = A^2 + 2AB + 2AC + B^2 + 2BC + C^2\]

First Layer

\[(A + B + C)^3 = A^3 + 3A^2B + 3A^2C + 3AB^2 + 6ABC + 3AC^2 + B^3 + 3B^2C + 3BC^2 + C^3\]

Second Layer

Third Layer
Level 0
Level 1
Level 2
Level 3
Level 4

Constant term: 1
Linear terms: 3
Quadratic terms: 6
Cubic terms: 10
Quartic terms: 15

\[ \begin{align*}
\text{4 terms} & : 4 \\
\text{10 terms} & : 10 \\
\text{20 terms} & : 20 \\
\text{35 terms} & : 35
\end{align*} \]
1. Let $U = \{a, b, c, d, e\}$, $S = \{b, c, d\}$, and $T = \{a, c, e\}$. List the elements of the following sets:
   (a) $S' \cap T$
   (b) $(S \cup T)'$

2. True or False
   a) $5 \in \{3, 5, 7\}$
   b) $\{1, 3\} \subseteq \{1, 2, 3\}$
   c) $0 \in \emptyset$
   d) $\emptyset \subseteq \{a, b, c\}$
   e) $1 \notin \{1\}$
   f) $\{b, c\} \subseteq \{b, c\}$
   g) If $A = \{1, 2, 2, 3\}$, then $n(A) = 4$.

3. Find the coefficient of $x^4y^3$ in the expansion of $(3x - 2y)^7$.

4. How many subsets of the set $\{a, b, c, d, e\}$ do not contain the letter $c$?
5. How many different tips could you leave in a restaurant if you had a nickel, a dime, a quarter, and a half-dollar?

(Yah, you’re pretty cheap regardless of what you leave, but this I just a math problem, OK!)

6. In a survey of 275 gang bangers, we asked them which weapons they liked best. It is found that 62 like atomic bombs, 91 like knives, 56 like guns, 28 like both atomic bombs and knives, 21 like both atomic bombs and guns, 25 like both knives and guns, and 12 of the gang bangers like all three weapon choices.

How many of the 275 gang bangers do not like any of these 3 weapon choices?

And how many like to use knives AND guns, but don’t use atomic bombs?

(HINT: FIRST DRAW THE VENN DIAGRAM BELOW, then ANSWER THE Qs above.)

7. The Choral Society and the Drama Club hold a joint party. Of the 60 people attending, 40 are members of the Choral Society and 30 are members of the Drama Club. How many are members of both groups?

(a) Do the problem using the inclusion-exclusion formula.

(b) Do the problem using a Venn diagram.
8. There are 9 pugs and 10 bulldogs at the *Annual Dogs With Old Man Faces Conference*.

(a) How many ways can we choose a team of 5 dogs if 3 must be pugs and 2 must be bulldogs?

(b) How many ways can we choose a team of 5 dogs if at least 1 must be a pug?

9. How many ways to get a full-house in a 5-card poker hand?

10. How many ways can the letters in the word STATISTIC be arranged (i.e., distinguishably different)?
11. How many different license plates are there if it must include 4 letters and 3 numbers (repetition allowed)?

12. How many different license plates are there if it must include 4 letters and 3 numbers (repetition not allowed)?

13. A fair coin is tossed 8 times, and each time the head or tail (H or T) is recorded.  
(a) How many possible outcomes are there when you toss a coin 8 times?

(b) How many ways can you get EXACTLY 4 heads?

(c) How many ways can you get at most 6 heads?

14. Hi everyone. Last question is the following: When is the Ch.5 math quiz? And will Pascal’s Triangle be given to you?
6.1 Experiments, Outcomes, Sample Spaces, & Events

Definitions:
1. An ______ is an activity with an observable result.
2. Each possible result is called an ______.
3. The ______ is the set of all possible outcomes.  
   (Note: We usually use the symbol \( S \) to denote this.)
4. An ______ is a subset of the sample space.  
   (We say an event \( E \) occurs when the outcome of an experiment is an element in the set \( E \).)

[Example] Consider the experiment of tossing a coin 3 times and observing the sequence of heads (H) and tails (T).
(a) Determine the sample space \( S \).
(b) Determine the event $E$, that we get exactly 2 heads.

$$E = \{ \}$$

[You finish]

(c) The probability that we get exactly 2 heads when flipping a coin 3 times is

$$\Pr(E) = \frac{n(E)}{n(S)} = \uparrow$$

[You finish]

---

Example 2: An experiment consists of rolling two dice (one red & one green) and observing the number on the uppermost face of each.

(a) Determine the sample space $S$. [You do!]

Note: Write all the outcomes below.
(b) Determine the event $E$ that the sum of the dice is 6.

$$E = \{ \}$$

[you finish]

(c) Determine the event $F$ that the numbers rolled are identical.

$$F = \{ \}$$

[you finish]

(d) Determine the event $G$ that the sum is at least 10.

$$G = \{ \}$$

[you finish]

**Definition** We say two events $A$ and $B$ in a sample space $S$ are **mutually exclusive** (or disjoint) if $A \cap B = \emptyset$. (i.e., they have no outcomes in common)

**Question:** Of the 3 events $E$, $F$, and $G$ above (in Example 2), which are mutually exclusive and which are not? And why? 

[you do!]
Example 3  An experiment consists of rearranging the letters in BANANA to form distinguishably different arrangements.

(a) Compute $n(S)$ where $S$ is the sample space.

[You do!]

(b) Let $E =$ event the "word" starts with B.

$F =$ event the "word" ends with A.

$G =$ event the "word" starts with A and ends with B.

Q1] Which are mutually exclusive?

Q2] Compute $n(E)$, $n(F)$, and $n(G)$?

Q3] Describe in words $E'$ and $G'$.

Q4] Describe $E \cap G$. 
In an experiment, a fair coin is tossed 15 times and the face that appears H for head and T for tails for each toss is recorded.

1. How many possible outcomes are there? (i.e., what is the size of the sample space?)

2. How many elements of the sample space will have exactly one head?

3. How many elements of the sample space will have an odd number of heads?
4. How many elements of the sample space will start and end with different faces and have a total of exactly three heads? (draw some examples first)

5. How many elements of the sample space will start or end (or both) with a head and have exactly four heads? (draw some examples first)
6. How many elements of the sample space will start or end (or both) with a head and have an adjacent triple or adjacent quadruple of heads and include a total of exactly four heads? (draw some examples first)
Is This Page Intentionally Left Blank for Your Doodling Delight? **Nope.** It is placed here to ensure that the new section on the next page begins with an odd-numbered page.
6.2 Assignment of Probabilities

(8 Inclusion-Exclusion Principle for Probabilities)

Definition: Suppose a sample space $S$ has a finite number of outcomes $s_1, s_2, \ldots, s_n$. To each outcome $s_i$ we associate its probability $p_i$. A probability distribution is a chart (or table) that gives these outcomes & probabilities.

Example 1

Roll a die once & observe the number rolled.

Sample Space $S = \{ 1, 2, 3, 4, 5, 6 \}$

<table>
<thead>
<tr>
<th>Outcome</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Example 2

Toss a coin four times & observe the number of heads.

<table>
<thead>
<tr>
<th>Outcome</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 heads</td>
<td></td>
</tr>
<tr>
<td>1 heads</td>
<td></td>
</tr>
<tr>
<td>2 heads</td>
<td></td>
</tr>
<tr>
<td>3 heads</td>
<td></td>
</tr>
<tr>
<td>4 heads</td>
<td></td>
</tr>
</tbody>
</table>

Q: What is the total # of observable sequences of $H^4$ and $T^4$ when flipping a coin four times?
Q: What are the sums of the values in the 2nd column of each table or previous two examples?

A:

Fact: [Two Fundamental Properties]

Given a sample space $S$ with finite # of outcomes $S_1, S_2, \ldots, S_n$ and corresponding probabilities $p_1, p_2, \ldots, p_n$. Then

I] Each probability $p_i$ is between 0 & 1 (exclusive).

II] $p_1 + p_2 + \ldots + p_n = \ldots$

[you finish]

Definition: Suppose $E$ is an event with outcomes $0_1, 0_2, \ldots, 0_n$. That is $E = \{0_1, 0_2, \ldots, 0_n\}$. Then the addition principle says $Pr(E) = Pr(0_1) + Pr(0_2) + \ldots + Pr(0_n)$.

Example 3] Roll a die once & observe the number rolled. So $S = \{1, 2, 3, 4, 5, 6\}$ is the sample space.

Let $E$ = event the die is odd

[you finish]

$\Pr(E) =$

[you finish] using addition principle
Inclusion-Exclusion Principle for Probabilities

Let $E$ and $F$ be events in a sample space $S$.

Then $Pr(E \cup F) =$

[You guess]

Hint: It is very similar to
the case of Inclusion-Exclusion
Principle you know from sets.

The Complement Rule for Probabilities

Let $E$ be an event in a sample space $S$.

Then $Pr(E) = 1 - Pr(E')$ & $Pr(E') = 1 - Pr(E)$

Example 4: Suppose $Pr(E) = .6$ and $Pr(F) = .5$
and $Pr(ENF) = .4$

a) Compute $Pr(E \cup F)$ using Inclusion-Exclusion Principle.
[You do!]

b) Compute $Pr(ENF')$ using a Venn-diagram approach.
[You do!]
Inclusion-Exclusion Principle for Probabilities
Worksheet

1. Experiment: Rearrange the letters in BANANA to form distinguishably different arrangements.
   Determine \( n(S) \)

2. Let \( E \) = the event a "word" starts with B
   Determine \( n(E) \)

3. Let \( F \) = the event a "word" ends with A
   Determine \( n(F) \)

4. What is \( Pr(E) \) and \( Pr(F) \)?
(5) What is the probability that either the event E or the event F occurs?
(Hint: Use Inclusion-Exclusion Principle)

37. Grades Joe feels that the probability of his getting an A in history is .7, the probability of getting an A in psychology is .8, and the probability of getting an A in history or psychology is .9. What is the probability that he will get an A in both subjects?

Use Venn diagram then use the Incl.-Excl. Formula.
Is This Page Intentionally Left Blank for Your Doodling Delight? **Nope.** It is placed here to ensure that the new section on the next page begins with an odd-numbered page.
A long-awaited definition:

Let $S$ be a sample space of equally-likely outcomes.
Let $E$ be an event in $S$ (i.e. $E \subseteq S$).

Then the probability that $E$ occurs is defined as

$$Pr(E) = \frac{n(E)}{n(S)}.$$

The Complement Rule

Let $E$ be an event and $E'$ be its complement. Then

$$Pr(E) = 1 - Pr(E').$$

Q: What is the probability of rolling at least a 2 in a single dice roll?

(Use Complement Rule)
An urn contains 10 blue balls and 4 red balls. A sample of 3 balls is selected at random.

1) What is \( n(S) \)?

2) What is the probability of selecting only red balls?

3) What is the probability of selecting only blue balls?

4) Probability of selecting at least 2 red balls?
What is the probability of getting a 3-of-a-kind hand in a 5-card poker hand?

**e.g.**

```
J J J 8 7 8

3-of-a-kind
```

Neither of these can be a J and they must also not be a 2-of-a-kind.
The Famous Birthday Problem

Mathematical Fact (which we will prove)
If there are at least 23 people in a room, then there is a more than 50% chance that at least 2 people have the same bday.

Note: For purposes of this problem ignore Feb 29th. So a year has 365 days.

Note: If 40 people, then 89% chance!
If 50 people, then 97% chance!

Q: Do the problem for a room of 10 people.
1. (3 points) Math 104 Students: To do this and the next problem, it will help to recall the inclusion exclusion principle (from Section 6.2). It states that if $E$ and $F$ are any events, then

$$P(E \cup F) = P(E) + P(F) - P(E \cap F).$$

In particular, if $E$ and $F$ are mutually exclusive, then

$$P(E \cup F) = P(E) + P(F),$$

The above is true since if two events are mutually exclusive then their intersection is the nullset and note that $P(\emptyset) = 0$.

Also it will help to know the complement rule (as shown in the brown box on pg.263). It states that if $E$ is any event, with its complement denoted $E'$, then

$$P(E) = 1 - P(E').$$

Now, bearing that in mind here is the problem:

If $P(A) = 0.45$, $P(B) = 0.55$ and $P(A \cap B) = 0.05$, find the following probabilities:

a) $P(A \cup B) = \text{_____}$

b) $P(A') = \text{_____}$

c) $P(B') = \text{_____}$

d) $P(A \cap B') = \text{_____}$ **Hint:** Make a Venn diagram like Example 11 on pgs.255-256 of Section 6.2.

Math 104 Students: Recalling how to shade Venn diagrams from Ch.5 is very helpful in the problem above and below. See pgs. 198-199 to get some refreshers.

e) $P((A \cap B)') = \text{_____}$
2. (3 points) Math 104 Students: This is like the previous problem however now the events are mutually exclusive.
If \( A \) and \( B \) are two mutually exclusive events with \( P(A) = 0.4 \) and \( P(B) = 0.5 \), find the following probabilities:
a) \( P(A \cap B) = \) 
b) \( P(A \cup B) = \) 
c) \( P(A') = \) 
d) \( P(B') = \) 
e) \( P((A \cup B)') = \) 
f) \( P(A \cap B') = \) 

3. (3 points) Math 104 Students: The following problem is an application of some of the essential ideas in the previous two problems.
Real estate ads suggest that 57% of homes for sale have garages, 30% have swimming pools, and 16% have both features.

What is the probability that a home for sale has
a) a pool or a garage?
Answer = _____ %
b) neither a pool nor a garage?
Answer = _____ %
c) a pool but no garage?
Answer = _____ %
1. (3 points) local/Library/ASU-topics/setProbability/events6.png

**Math 104 Students:** The problems in this WeBWorK set are probability problems. All will rely on your understanding of the counting principles you learned in Ch.5. So this is an excellent review for some of the material that may appear on Exam 2.

A fair coin is tossed 8 times. What is the probability that:

a) Exactly 4 heads appear? _____

b) At least two heads appear? _____

**Hint:** It might be easier to calculate the probability that this event does NOT happen, and subtract that probability from 1. (See the complement rule on pg.263).

c) At most 5 heads appear? _____

**Hint:** You might want to use the addition principle on pg.253. That is, the probability that at most 5 heads appear equals the following:

\[ P(0 \text{ heads appear}) + P(1 \text{ heads appear}) + \cdots + P(5 \text{ heads appear}). \]
2. (3 points) A box contains 22 yellow, 25 green and 33 red jelly beans. If 10 jelly beans are selected at random, what is the probability that:

a) Exactly 7 are yellow? _________ (See hint below. Actually, I give the full solution and answer.)
b) Exactly 7 are yellow and exactly 2 are green? _________ (See hint below.)
c) At least one is yellow? _________ (See hint below.)
Two Interest Formulas

1) Simple Interest given
   \[ F = (1 + nr) \cdot P \]

2) Compound Interest given
   \[ F = (1 + i)^n \cdot P \]

- \( n \) = # of years
- \( r \) = rate (in decimal)
- \( P \) = initial investment
- \( P \) = initial investment
- \( r \) = rate (in decimal)
- \( m \) = # of times compounded per year
- \( c = \frac{r}{m} \) (interest rate per period)
- \( n = \text{total # of interest periods} \) (usually \( \text{# of years} \times m \))

Suppose you need $10,000 in exactly 2\frac{3}{4} \text{ years from now}. How much do you need to invest now if the bank pays 5\% \text{ interest compounded monthly}?

A: \[ P = \]
   \[ r = \]
   \[ m = \]
   \[ c = \]
   \[ n = \]
   \[ F = \]
Q: Compute the future value of an investment of $420 after 5 years into an account paying a simple interest of 8%.

\[ P = \quad F = \quad n = \quad r = \]

Q: Is it a "better deal" to go to a bank that pays 8% interest compounded annually rather than paying simple interest. Use same \( P = \$420 \) and 5 years as above.

\[ P = \quad F = \quad r = \quad m = \quad i = \quad n = \]
Q. How many years are required for $500 to grow to $800 at 1.5% simple interest?

Q. How many years are required for $100 to double if deposited at 3.2% interest compounded quarterly?
Consider the following two options

(A) 4% simple interest
(B) 4% interest compounded daily

After how many years will option B outperform option A by at least $100?  

Answer: 11 yrs

(If the initial deposit is $1000.)
10.2 Annuities

Def. An annuity is a sequence of equal payments, made at regular intervals of time.

Two types of Annuities

I] Increasing Annuity
eg. You make equal payments to a bank to generate a large sum of $ in the future.
book eg. Putting $1 aside each yr for your newborn child to save for your college educ. 18 yrs from now.

II] Decreasing Annuity
eg. The bank will make equal payments to you in order to pay back the sum of $ you currently deposit.
book eg. You win the lotto & decide not to work for 5 yrs but want $5000 at end of each month.

Def. The future value of an increasing annuity of n equal payments is the value of the annuity after the n-th payment.

The Incr. Annuity Formula
Suppose an increasing annuity consists of n payments of $ R dollars each, deposited at the ends of consecutive interest periods into an acct with interest compounded at a rate of i per period. Then the future value F of the annuity is

\[ F = \frac{(1+i)^n - 1}{i} \cdot R \]
Eg) Suppose you have a newborn daughter & want to save for her college education 18 yrs from now. So you deposit $100 at the end of each month into an acct. paying 6% interest compounded monthly.

Q: 18 years from now, how much is in the acct?

\[ R = ? \quad i = \frac{r}{m} = ? \quad n = ? \]

Q: After 18 yrs, how much interest did you earn? (i.e. subtract your total payments from your future value).
Def. The present value of a decreasing annuity is the amount you need to deposit in order for you to receive a desired sequence of annuity payments $R$ and leave a balance of zero at the end of the term.

If interest is compounded at a rate $i = \frac{r}{m}$ per interest period, then

$$P = \frac{1 - (1+i)^{-n}}{i} \cdot R$$

E.g. You won the lotto! You decide to quit your job and not work for 5 years. You need $50000/mo. to live comfy. If bank pays 6% interest compd monthly, then what must you deposit now? [You do!]

$$R = \ ?$$

$$P = \ ?$$

$$m = \ ?$$

$$r = \ ?$$

$$i = \ ?$$

$$n = \ ?$$

$$P = \frac{1 - (1 + 0.005)^{-60}}{0.005} (500)$$
If time is left after the worksheet,
do problem similar to #7 in 10.1 HW.

Q: You invest 400 dollars into an account paying simple
8.4% interest, and B dollars "" "" 7.1% interest compd. quarterly. After 6 years
the accounts have the same balance.

What is B?

Recall Simple interest formula    \[ F = (1 + nr) \cdot P \]
& Compound " "    \[ F = (1 + r)^n \cdot P \]
Introduction to Ch. 10.3 on Amortization

Topic: The mathematics of paying off loans.

Defn: "Amortization" is the process of paying off a loan.

Example: You take out a loan to buy a TV set for $563. Loan interest rate is 12% compounded monthly.

Q: What is the monthly payment if you need to pay it off in 5 months? [You do!]

<table>
<thead>
<tr>
<th>Payment number</th>
<th>Amount</th>
<th>Interest</th>
<th>Applied to principal</th>
<th>Unpaid balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$116</td>
<td>$5.63</td>
<td>$110.37</td>
<td>$452.63</td>
</tr>
<tr>
<td>2</td>
<td>116</td>
<td>4.53</td>
<td>111.47</td>
<td>341.16</td>
</tr>
<tr>
<td>3</td>
<td>116</td>
<td>3.41</td>
<td>112.59</td>
<td>228.57</td>
</tr>
<tr>
<td>4</td>
<td>116</td>
<td>2.29</td>
<td>113.71</td>
<td>114.85</td>
</tr>
<tr>
<td>5</td>
<td>116</td>
<td>1.15</td>
<td>114.85</td>
<td>0.00</td>
</tr>
</tbody>
</table>
1) Suppose you won the lottery. You want to not work for 5 years but get paid $5000/month to live comfortably. If the bank pays 6% interest compounded monthly, then what must you deposit now?

2) How much interest did you earn at the end of the term of the annuity?

Answer = [What the bank paid you in those five years] - [What you initially gave the bank at time = 0]
3) Calculate the rent of a decreasing annuity at 8% compounded quarterly with payments made every quarter-year for 7 years and present value of $100,000.

4) Is this a decreasing or increasing annuity problem?
   a) Suppose you deposit $1,700 each year into an IRA (Individual Retirement Account) earning 13% compounded annually for 18 years. What is the value of the IRA after 18 years?

   b) How much can you withdraw each year for the next 25 years at 13% compounded annually?
Is This Page Intentionally Left Blank for Your Doodling Delight? **Nope.** It is placed here to ensure that the new section on the next page begins with an odd-numbered page.
Here is something similar to the 1st WeBWorK problem in Section 10.3:

Mr. and Mrs. BluGold have just purchased a $150000 house and made a down payment of $11000. They can amortize the balance at 11% over 30 years.

a) What are the monthly payments?
Answer = $

b) How much interest will be paid?
Answer = $

c) What is their equity after 5 years?
Answer = $

d) What is their equity after 25 years?
Answer = $

HINT:

a) We first find the amount of the loan as \( P = $150000 - $11000 = $139000 \). Since this is an amortized loan we use the Present Value formula as given in the decreased annuity formula (recall Section 10.2 pg.443)

\[
P = \frac{1 - (1 + i)^{-n}}{i} R
\]

to find the monthly payments \( R \).

b) The total interest paid is the total amount paid minus the amount of the loan.

c) The equity on the house after 5 years is: (Down payment) + (Amount paid on loan). The amount paid on the loan is:
(original loan amount) - (Present value after 5 years).
After 5 years there are (25)12 payments left. Thus to find the present value we use the present value formula with \( n = (25)12 = 300 \).

d) We reason similarly to part c) with \( n = 5(12) = 60 \).
SOLUTION:

a) We first find the amount of the loan as \( P = $150000 - $11000 = $139000 \).
Since this is an amortized loan we use the Present Value formula
\[
P = \frac{1 - (1 + i)^{-n}}{i} R
\]
with
\( P = 139000, \ i = \frac{0.11}{12}, \ n = 360 \) to find the monthly payments
\[
R = \frac{139000 \left( \frac{0.11}{12} \right)}{1 - (1 + \frac{0.11}{12})^{-360}} = 1323.73.
\]

b) The total interest paid is the total amount paid minus the amount of the loan. The total amount
paid is
\[
1323.73(30)(12) = 476542.63,
\]
thus the interest is
\[
I = 476542.63 - 139000 = 337542.63.
\]

WARNING! BE CAREFUL MATH 104 STUDENTS: Don't use the \( R \)-value like I wrote in
the line above, use the REAL \( R \)-value (i.e., the one without rounding, even though the \( R \)-value
you put in part (a) is the rounded value). Sorry.

c) We first find the Present Value after 5 years (this is the amount on the principal of the loan
we still owe).
After 5 years there are \( n = (25)(12) = 300 \) payments left, thus the present value (again,
using the decreased annuity formula) is
\[
P = 1323.73 \frac{1 - (1 + \frac{0.11}{12})^{-300}}{\frac{0.11}{12}} = 135058.86.
\]
Hence, the amount already paid on the loan is
\( 139000 - 135058.86 = 3941.14 \).
The equity on the house after 5 years is:
\[
\text{Equity} = (\text{Down payment}) + (\text{Amount paid on loan}) = 11000 + 3941.14 = 14941.14.
\]
d) We first find the Present Value after 25 years (this is the amount on the principal of the loan
we still owe).
After 25 years there are \( n = (5)(12) = 60 \) payments left, thus the present value is
\[
P = 1323.73 \frac{1 - (1 + \frac{0.11}{12})^{-60}}{\frac{0.11}{12}} = 60882.34.
\]
Hence, the amount already paid on the loan is
\( 139000 - 60882.34 = 78117.66 \).
The equity on the house after 25 years is:
\[
\text{Equity} = (\text{Down payment}) + (\text{Amount paid on loan}) = 11000 + 78117.66 = 89117.66.
\]
GROUP WORK

Suppose you purchase a $100000 house using a down payment of $12000. They can amortize the balance at 5% over 20 years.

a) What is their monthly payment?
Answer = $

b) What is the total interest paid?
Answer = $

c) What is the equity after 5 years?
Answer = $

d) What is the equity after 15 years?
Answer = $

A couple of lovebirds have decided to purchase a $90,000 nest using a down payment of $17,000. They can amortize the balance at 7% over 25 years.

a) What is their monthly payment?

b) What is the total interest paid?
c) What is the equity after 5 years?

d) What is the equity after 20 years?
Defn: A sequence is an ordered collection of objects. e.g., 2, 4, 6, 8, 

Convention: 
- \( a_0 \) = the first element 
- \( a_1 \) = the 2nd " 
- \( a_2 \) = the 3rd " etc. 

Alt'ly, your book uses \( y_0, y_1, y_2, \ldots \) 

Defn: A difference equation is an equation of the form \[ y_n = a \cdot y_{n-1} + b \] where \( a \) & \( b \) are specific numbers. 

[You do!] Let \( y_0 = -2 \). Consider the difference equation \( y_n = 2y_{n-1} + 3 \). Write the first 5 terms. Can you guess a "closed form" for \( y_n \)?
Example 1

\[ y_0, y_1, y_2, y_3, y_4, \ldots, y_n, \ldots, y_{100} \]

2, 4, 6, 8, 10, \ldots

What is the difference equation?
& initial value

Example 2

\[ y_0, y_1, y_2, y_3, y_4, \ldots, y_n, \ldots, y_{100} \]

1, 3, 5, 7, 9, \ldots

Difference Equation?
& initial value

Example 3

\[ y_0, y_1, y_2, y_3, y_4, \ldots, y_n, \ldots, y_{100} \]

1, 2, 4, 8, 16, \ldots

Difference Equation?
& initial value

Example 4: If \( y_n = 2y_{n-1} + 1 \) is the difference equation with initial value \( y_0 = 0 \), write the first 5 terms.

Guess a closed form.
Def: A closed form for the difference equation \( y_n = ay_{n-1} + b \) with initial condition \( y_0 \) is a formula for \( y_n \) that does not rely on knowing \( y_{n-1} \).

\[ \text{Two Formulas [for closed forms for } y_n=ay_{n-1}+b \text{ with init. cond. } y_0] \]

\((\star)\) If \( a \neq 1 \), then
\[
y_n = \frac{b}{1-a} + \left(y_0 - \frac{b}{1-a}\right)a^n.
\]

\((\star\star)\) If \( a = 1 \), then \( y_n = y_0 + bn \).

[You do!] Find a closed form for \( y_n = 2y_{n-1} + 3 \) with initial condition \( y_0 = -2 \).
Q4 Example) If $a_n = 6n - 10$, then what is the 9th term of the sequence?

[Note: we are not looking for $a_9$, why?]

Hint: first term = $a_0$. Compute it & $a_1$ for starters.

Q5 Example) Suppose $a_n = -2a_{n-1} + 3a_{n-2}$ & $a_5 = -118$ & $a_4 = 44$. Find $a_1, a_2, & a_3$. 
Simple & Compound Interest & Depreciation

Via difference equations

Recall the \((\star)\) & \((\star\star)\) formulas

let \( y_n = a \cdot y_{n-1} + b \) where \( a \) & \( b \) are numbers & an initial condition \( y_0 \) is given.

Then the difference equation has "solution" (i.e., closed form)

If \( a \neq 1 \), then

\[
y_n = \]

If \( a = 1 \), then

\[
y_n = \]

Recall If interest is compounded \( m \) times per year & the annual interest rate is \( r \), the the interest rate per period is \( \]
Compound Interest Example:

Suppose an account initially contains $40, & earns 6% interest \(^{\text{compounded}}\) annually. Determine a formula that describes the \(n^{\text{th}}\) year's balance based on the \((n-1)^{\text{st}}\) year's balance.

<table>
<thead>
<tr>
<th>year</th>
<th>balance</th>
<th>interest</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(y_0 = 40)</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>(y_1 = 40 + 2.40)</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>(\ldots)</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>(\ldots)</td>
<td></td>
</tr>
</tbody>
</table>

Q: What is the recursion formula (i.e., difference equation) that describes the \(n^{\text{th}}\) year's balance?

Hint: Let \(y_n\) = balance at the end of the \(n^{\text{th}}\) interest period. [In this case, "at end of \(n^{\text{th}}\) year".]
**Simple Interest Example:**

Suppose an account initially contains $40, & earns 6% simple interest annually. Determine a formula that describes the nth year's balance based on the (n-1)st year's balance.

<table>
<thead>
<tr>
<th>Year</th>
<th>Balance</th>
<th>Interest</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$40</td>
<td>$(0.06) \times 40 = 2.40$</td>
</tr>
<tr>
<td>1</td>
<td>$40 + 2.40$</td>
<td>$(0.06) \times 40 = 2.40$</td>
</tr>
<tr>
<td>2</td>
<td>$Y_2 = \ldots$</td>
<td>\textbf{Remember, this is NOT compounded!}</td>
</tr>
<tr>
<td>3</td>
<td>$Y_3 = \ldots$</td>
<td>\textbf{Remember, this is NOT compounded!}</td>
</tr>
</tbody>
</table>

Q: What is the recursion formula that describes the nth year's balance?
Finding Closed Forms

1. Solve the difference equation you got on pg. 2. This is a compound interest formula. Why?

2. Solve the difference equation you got on pg. 3. This is a simple interest formula. Why?
Sample Problems

Consider the difference equation \( y_n = -3y_{n-1} + 2 \) with initial condition \( y_0 = 4 \).

1. Write down the first five terms.
   (Can you guess the 6th term? or a closed form?)

2. Find the closed form. What is \( y_{10} \)?

Consider the difference equation \( y_n = y_{n-1} + 6 \) with initial condition \( y_0 = 2 \). Do the same two questions above.
21. **Depreciation Modeling** A condo is purchased for $50,000 and depreciated over a 25-year period. Let $y_n$ be the (undepreciated) value of the condo after $n$ years. Determine and solve the difference equation for $y_n$, assuming straight-line depreciation (i.e., each year the condo depreciates by $\frac{1}{25}$ of its original value).

22. **Depreciation Modeling** Refer to Exercise 21. Determine and solve the difference equation for $y_n$, assuming the double-declining balance method of depreciation (i.e., each year the condo depreciates by $\frac{2}{25}$ of its value at the beginning of that year).
Difference equations can come up while on a date.

I almost got it! Let’s see, if $y_0 = 5$ and $y_n = 2y_{n-1} - 4$, then I think the closed form is $y_n = \sqrt{n} + \pi$? That can’t be right! Darn it! Hold on dear, I’ll get it right.

Oh, maybe this is thing you use the quadratic formula to solve it?

Waiter, can we have the check? And can you kindly tell my lame date that if $y_n = ay_{n-1} + b$ with initial condition $y_0$, then a closed form is easily found using the star-formula.

This is the LAST TIME that I go on a blind date with someone who has never taken Prof aBa’s Math 104.

[TO DO!] Let’s solve this on the next page.

**SOLUTION:** (don’t peek! That’s why it’s super small font!)
The recurrence formula gives the following sequence: 5, 6, 8, 12, 20, 36, … . Using the formula that the woman says above, we can deduce that a closed form is given by $y_n = 4 + 2^n$. 

Page 171 of 217
RECALL the Star and Star-Star formulas:

Given \( y_n = ay_{n-1} + b \) with initial condition \( y_0 \), then the closed form solution is

- if \( a \neq 1 \), use \( y_n = \frac{b}{1-a} + (y_0 - \frac{b}{1-a})a^n \).

- if \( a = 1 \), use \( y_n = y_0 + bn \).

(QUESTION 1) Consider the difference equation \( y_n = 2y_{n-1} - 4 \) with initial condition \( y_0 = 5 \). Use the difference equation to find the first 5 terms and then give a closed form for difference equation. Verify that your closed formula yields the first 5 terms.
(QUESTION 2) Consider the difference equation $y_n = \frac{1}{2} y_{n-1} - 1$ with initial condition $y_0 = 10$. Write out the first 5 terms and then give a closed form for difference equation. What is $y_{17}$? As $n$ approaches infinity, can you determine what $y_n$ approaches?

(QUESTION 3) Consider the difference equation $y_n = y_{n-1} + 3$ with initial condition $y_0 = 2$. Write out the first 5 terms and then give a closed form for difference equation. What is $y_{17}$? As $n$ approaches infinity, can you determine what $y_n$ approaches?
(QUESTION 4) Suppose you deposit $600 into a bank charging 9% compounded monthly. Let $y_n$ equal the amount at the end of $n$ months. Write a difference equation that models this compound interest problem. Then find an explicit formula for $y_n$. How much is in the bank after 2 years?

(QUESTION 5) Suppose you deposit $600 into a bank charging 9% simple interest. Let $y_n$ equal the amount at the end of $n$ years. Write a difference equation that models this compound interest problem. Then find an explicit formula for $y_n$. How much is in the bank after 2 years?
(QUESTION 5) [Automobile Depreciation] A rule of thumb states that cars in personal use depreciate by 15% each year. Suppose that a new car is purchased for $20,000. Let $y_n$ be the value of the car after $n$ years.

a) Find a difference equation satisfied by $y_n$.

b) Solve the difference equation.

c) How much will the car be worth after 5 years.
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Some Math 104 Exam 1 Prep Questions

Multiple Choice
Identify the choice that best completes the statement or answers the question.

1. Find an equation of the line that passes through the points.
   \((-3, -5)\) and \((1, -6)\)
   a. \(y = \frac{1}{4}x - \frac{23}{4}\)
   b. \(y = -\frac{1}{4}x - \frac{23}{4}\)
   c. \(y = \frac{1}{2}x - \frac{13}{2}\)
   d. \(y = \frac{1}{2}x - \frac{13}{2}\)
   e. \(y = -\frac{1}{2}x - \frac{23}{4}\)

2. Find the constants \(m\) and \(b\) in the linear function \(f(x) = mx + b\) so that \(f(1) = 3\) and the straight line represented by \(f\) has slope 1.
   a. \(m = -1, b = 5\)
   b. \(m = -2, b = 5\)
   c. \(m = -1, b = 4\)
   d. \(m = -2, b = 4\)

3. The quantity demanded for a certain brand of portable CD players is 200 units when the unit price is set at $100. The quantity demanded is 1,200 units when the unit price is $50. Find the demand equation.
   a. \(p = -0.05x + 110\)
   b. \(p = -0.04x + 90\)
   c. \(p = -0.05x + 90\)
   d. \(p = -0.04x + 110\)

4. Consider the supply equation \(p = 2x + 11\), where \(x\) is the quantity supplied in units of 1,000 and \(p\) is the unit price in dollars. Determine the number of units of the commodity the supplier will make available in the market at the given unit prices \(p = 19\).
   a. 3,700
   b. 4,000
   c. 3,100
   d. 4,300
   e. 3,400
5. For the pair of supply-and-demand equations, where \( x \) represents the quantity demanded in units of 1,000 and \( p \) is the unit price in dollars, find the equilibrium quantity and the equilibrium price.

\[
p = -0.3x + 13.8 \quad \text{and} \quad p = 0.15x + 6.6
\]

a. equilibrium quantity 16,000 units; equilibrium price $9
b. equilibrium quantity 3,200 units; equilibrium price $17
c. equilibrium quantity 16,000 units; equilibrium price $17
d. equilibrium quantity 3,200 units; equilibrium price $9

6. Solve the linear system of equations

\[
\begin{align*}
2x - 6y &= 7 \\
5x + 2y &= 10
\end{align*}
\]

a. Unique solution: \((5, 1)\)
b. Unique solution: \(\left(\frac{37}{17}, \frac{-15}{34}\right)\)
c. Infinitely many solutions; \((t, 8t + 3)\)
d. No solution
7. Row reduce the matrix until there are zeros ABOVE and BELOW the bolded matrix entry.

\[
\begin{bmatrix}
0 & 2 & 3 & 3 \\
3 & 3 & 0 & 3 \\
5 & 5 & 2 & -4
\end{bmatrix}
\]

\[
\begin{bmatrix}
-9 & -7 & 0 & -6 \\
3 & 3 & 1 & 3 \\
-1 & -1 & 0 & -10
\end{bmatrix}
\]

\[
\begin{bmatrix}
-9 & -7 & 0 & 3 \\
3 & 3 & 1 & 3 \\
-1 & -1 & 0 & -4
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & -7 & 0 & 3 \\
3 & 3 & 1 & 3 \\
-1 & -1 & 0 & -4
\end{bmatrix}
\]

\[
\begin{bmatrix}
-9 & 2 & 0 & -6 \\
3 & 3 & 1 & 3 \\
-1 & -1 & 0 & -10
\end{bmatrix}
\]

\[
\begin{bmatrix}
-9 & 2 & 0 & -6 \\
3 & 3 & 1 & 3 \\
-1 & 5 & 0 & -10
\end{bmatrix}
\]

8. The total number of passengers riding a certain city bus during the morning shift is 700. If the child's fare is $1.15, the adult fare is $5.50, and the total revenue from the fares in the morning shift is 245, how many children and how many adults rode the bus during the morning shift?

- a. 300 children and 400 adults
- b. 360 children and 340 adults
- c. 460 children and 240 adults
- d. 300 children and 460 adults
- e. 400 children and 300 adults
9. Compute the product.

$$\begin{bmatrix} 3 & 8 \\ 9 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

a. $$\begin{bmatrix} -5 \\ 13 \end{bmatrix}$$

b. $$\begin{bmatrix} 1 \\ 12 \end{bmatrix}$$

c. $$\begin{bmatrix} -5 \\ 9 \end{bmatrix}$$

d. $$\begin{bmatrix} -8 \\ 9 \end{bmatrix}$$

10. Compute the indicated product.

$$\begin{bmatrix} 1 & 1 & -4 & 0 \\ 3 & -2 & -2 & 2 \\ -2 & 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 2 & 5 \\ 2 & -2 \\ 0 & -6 \end{bmatrix}$$

a. $$\begin{bmatrix} -3 & 11 \\ 1 & -24 \\ 0 & 13 \end{bmatrix}$$

b. $$\begin{bmatrix} 3 & 1 & 0 \\ 11 & 24 & -13 \end{bmatrix}$$

c. $$\begin{bmatrix} 3 & 11 \\ 1 & -24 \\ -1 & 13 \\ -3 & 11 \end{bmatrix}$$

d. $$\begin{bmatrix} 1 & -24 \\ 5 & 13 \end{bmatrix}$$

e. the problem has no solution
11. Write the system of linear equations in the matrix form: \( AX = B \). For ERROR credit, show your work by writing down (separately) what the matrices \( A \), \( X \), and \( B \) are.

\[
\begin{align*}
7x - 2y + 4z &= 9 \\
3y - 7z &= 4 \\
x - y + 2z &= 5
\end{align*}
\]

a. \[
\begin{bmatrix}
7 & -2 & 4 \\
0 & 3 & -7 \\
1 & -1 & 2
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
= 
\begin{bmatrix}
9 \\
4 \\
5
\end{bmatrix}
\]

b. \[
\begin{bmatrix}
0 & 7 & -3 \\
1 & -1 & 2 \\
7 & -3 & 7
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
= 
\begin{bmatrix}
11 \\
0 \\
5
\end{bmatrix}
\]

c. \[
\begin{bmatrix}
0 & 2 & -4 \\
1 & -1 & 2 \\
4 & -2 & 7
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
= 
\begin{bmatrix}
11 \\
4 \\
3
\end{bmatrix}
\]

d. \[
\begin{bmatrix}
0 & 3 & -7 \\
1 & -1 & 2
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
= 
\begin{bmatrix}
9 \\
5
\end{bmatrix}
\]

12. Writing the system of equations as a matrix equation solve the system of equations by using the inverse of the coefficient matrix.

\[
\begin{align*}
x + 4y &= 11 \\
4x - y &= -1
\end{align*}
\]

a. \( x = 7, y = -4 \)

b. \( x = \frac{22}{7}, y = \frac{31}{7} \)

c. \( x = 15, y = 11 \)

d. \( x = \frac{7}{17}, y = \frac{45}{17} \)

e. The system has no solution.
13. Determine graphically the solution set for the following system of inequalities. **NOTE:** This problem is **SHADING** the feasible set (which is different than what we did in the textbook and on the chalkboard). So the **SHADED PART IS THE FEASIBLE SET!!!**

\[
\begin{align*}
2x - 7y &\geq -24 \\
x + 3y &\geq 5 \\
x &\geq 0 \\
y &\geq 0
\end{align*}
\]
14. Find the inverse of the given matrix, if it exists. Verify your answer.

\[
\begin{bmatrix}
3 & 1 \\
5 & 2
\end{bmatrix}
\]

a. \[
\begin{bmatrix}
2 & 5 \\
-1 & 3
\end{bmatrix}
\]

b. \[
\begin{bmatrix}
2 & -1 \\
-5 & 3
\end{bmatrix}
\]

c. \[
\begin{bmatrix}
1 & -3 \\
-2 & 5
\end{bmatrix}
\]

d. \[
\begin{bmatrix}
-3 & 1 \\
5 & -2
\end{bmatrix}
\]

e. The inverse matrix does not exist.

15. Find the inverse of the given matrix, if it exists. Verify your answer.

\[
\begin{bmatrix}
2 & 1 \\
1 & 1
\end{bmatrix}
\]

a. \[
\begin{bmatrix}
1 & -1 \\
-1 & 2
\end{bmatrix}
\]

b. \[
\begin{bmatrix}
1 & -2 \\
-1 & 1
\end{bmatrix}
\]

c. \[
\begin{bmatrix}
1 & 1 \\
1 & 2
\end{bmatrix}
\]

d. \[
\begin{bmatrix}
-2 & 1 \\
1 & -1
\end{bmatrix}
\]

e. The inverse matrix does not exist.
16. Find the inverse of the matrix, if it exists. **NOTE:** Recall that I did not teach you how the Ch.2.5 technique of finding a 3x3 inverse. We only know how to get the 2x2 inverse. However, there is a way to find which of the four below is the inverse of the given matrix. What is that method?

\[
\begin{bmatrix}
2 & -3 & -3 \\
0 & 0 & -1 \\
1 & -2 & 1 \\
\end{bmatrix}
\]

a. \[
\begin{bmatrix}
2 & -9 & -3 \\
1 & -5 & -2 \\
0 & -1 & 0 \\
\end{bmatrix}
\]

b. \[
\begin{bmatrix}
1 & -5 & -2 \\
0 & -1 & 1 \\
2 & -5 & -3 \\
\end{bmatrix}
\]

c. \[
\begin{bmatrix}
1 & 6 & -2 \\
0 & -1 & 0 \\
2 & -6 & -6 \\
\end{bmatrix}
\]

d. \[
\begin{bmatrix}
1 & -5 & -3 \\
0 & -1 & 1 \\
\end{bmatrix}
\]

e. the inverse does not exist

17. The relationship between temperature measured in the Celsius scale and the Fahrenheit scale is linear. The freezing point is 0°C and 32°F, and the boiling point is 100°C and 212°F, respectively. Find \( F \) as a function of \( C \) and use this formula to determine the temperature in Fahrenheit corresponding to temperature of 50°C.

**NOTE:** Like all the other multiple-choice, you must show your work for full credit. In particular, this one asks you to find \( F \) as a function of \( C \) (so you must write down this function), using the given information above.

a. 125°F
b. 122°F
c. 121°F
d. 124°F
18. Cowling's rule is a method for calculating pediatric drug dosages. If $a$ denotes the adult dosage (in milligrams) and if $t$ is the child's age (in years), then the child's dosage is given by

$$D(t) = \left(\frac{t+1}{24}\right)a$$

If the adult dose of a drug is 520 mg, how much should a 3 year-old child receive?

a. 81.27 milligrams  
b. 84.87 milligrams  
c. 83.07 milligrams  
d. 86.67 milligrams  
e. 79.47 milligrams

19. If the line passing through the points $(a,1)$ and $(5,6)$ is parallel to the line passing through the points $(2,9)$ and $(a+2,3)$, what is the value of $a$?

$a =$________

20. Assume that a certain commodity's demand equation has the form $p = ax + b$, where $x$ is the quantity demanded and $p$ is the unit price in dollars. Suppose the quantity demanded is 3,000 units when the unit price is $17.00 and 8,000 when the unit price is $12.00. What is the quantity demanded when the unit price is $16.50?

_______ units

21. The annual sales of Crimson Drug Store are expected to be given by $S = 2.4 + 0.5t$ million dollars $t$ years from now, whereas the annual sales of Cambridge Drug Store are expected to be given by $S = 1.5 + 0.8t$ million dollars $t$ yr from now. When will Cambridge's annual sales first surpass Crimson's annual sales?

Please round your answer to one decimal place if necessary.

$t =$________ yr

22. Determine the value of $k$ for which the system of linear equations has no solution.

$$\begin{cases} 2x - y = 3 \\ 6x + ky = 7 \end{cases}$$

$k =$________
23. A theater has a seating capacity of 300 and charges $2 for children, $4 for students, and $5 for adults. At a certain screening with full attendance, there were half as many adults as children and students combined. The receipts totaled $1,100. How many children attended the show?

_______ children

24. Cindy regularly makes long-distance phone calls to three foreign cities: London, Tokyo, and Hong Kong. The matrices A and B give the lengths (in minutes) of her calls during peak and nonpeak hours, respectively, to each of these three cities during the month of June.

\[
A = \begin{bmatrix}
70 & 50 & 30 \\
\end{bmatrix} \quad B = \begin{bmatrix}
320 & 140 & 260 \\
\end{bmatrix}
\]

The costs for the calls (in dollars per minute) for the peak and nonpeak periods in the month in question are given, respectively, by the matrices

\[
C = \begin{bmatrix}
.35 \\
.42 \\
.48 \\
\end{bmatrix} \quad D = \begin{bmatrix}
.24 \\
.32 \\
.35 \\
\end{bmatrix}
\]

Compute the matrix \(AC + BD\). Give the answer to one decimal place.

Matching

*Match each graph below with the letter of the corresponding slope description.*

a. The slope of the line is zero.
b. The slope of the line is positive, and its y-intercept is negative.
c. The slope of the line is negative, and its x-intercept is negative.

____ 25.
28. Determine whether the lines through the given pairs of points are parallel. Answer yes or no.

\[ A (1, -2), B (-2, -8) \text{ and } C (1, 1), D (-1, 5) \]

29. The demand equation for a certain brand of GPS Navigator is \( x + 4p - 660 = 0 \), where \( x \) is the quantity demanded per week and \( p \) is the wholesale unit price in dollars. The supply equation is

\[ x - 15p + 1050 = 0 \]

where \( x \) is the quantity the supplier will make available in the market when the wholesale price is \( p \) dollars each. Find the equilibrium quantity and the equilibrium price for the GPS Navigators.

equilibrium quantity \( \underline{\text{units}} \)

equilibrium price \( \underline{\text{dollars}} \)

30. Write the augmented matrix corresponding to the given system of equations. Recall, that the augmented matrix is simply the matrix corresponding to the original system of linear equations. Do NOT do any row-reducing.

\[
\begin{align*}
6x + 3y &= 0 \\
x - y + 3z &= 6 \\
3y - 6z &= 3
\end{align*}
\]

Add to this: Find equation of the line which is perpendicular to the line \( \overline{AB} \) & goes through the point \( (2,8) \)
31. Indicate whether the matrix is in row-reduced form. Answer yes or no.
\[
\begin{bmatrix}
1 & 0 & 0 & 3 \\
0 & 1 & 0 & 9 \\
0 & 0 & 1 & 5
\end{bmatrix}
\]

32. Given that the augmented matrix in row-reduced form is equivalent to the augmented matrix of a system of linear equations, determine whether the system has a solution and find the solution or solutions to the system, if they exist. If not, answer no solution.
\[
\begin{bmatrix}
1 & 0 & 0 & -5 \\
0 & 1 & 0 & -1 \\
0 & 0 & 1 & 6
\end{bmatrix}
\]

33. Given that the augmented matrix in row reduced form is equivalent to the augmented matrix of a system of linear equations, determine whether the system has a solution.
\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 8 \\
0 & 1 & 0 & 0 & -4 \\
0 & 0 & 1 & 1 & 3 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

(Yes/no)

Find the solution or solutions to the system, if they exist.
\[
x = \\
y = \\
w = \\
z =
\]

If you think there are infinitely many solutions, write your answer as \( z = z \) and \( w \) as a function of \( z \).
34. Find the sizes of $A$, $B$, $C$ and $D$.

\[
A = \begin{bmatrix}
3 & -1 & 2 \\
0 & 1 & 4 \\
3 & 2 & 1 \\
-1 & 1 & 8
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
1 & 0 & 3 & 4 & 5
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
1 \\
3 \\
-2 \\
0
\end{bmatrix}
\]

\[
D = \begin{bmatrix}
2 & -3 & 9 & -4 \\
-11 & 2 & 6 & 7 \\
6 & 0 & 2 & 9 \\
5 & 1 & 5 & -8
\end{bmatrix}
\]

$A$: _______ $\times$ _______$B$: _______ $\times$ _______$C$: _______ $\times$ _______$D$: _______ $\times$ _______$;

Additional Q:
Which of the matrices can be multiplied together?
35. Find $a_{14}, a_{21}, a_{31}$ and $a_{43}$.

\[
A = \begin{bmatrix}
7 & -9 & 2 & -6 \\
-11 & 11 & 11 & 6 \\
10 & 11 & 7 & 11 \\
1 & 6 & 7 & -2
\end{bmatrix}
\]

$a_{14} =$

$a_{21} =$

$a_{31} =$

$a_{43} =$

36. Compute the product.

\[
\begin{bmatrix}
2 & 1 & 6 \\
-1 & 6 & 5
\end{bmatrix}
\begin{bmatrix}
6 \\
1 \\
-2
\end{bmatrix}
\]

37. Compute the product.

\[
\begin{bmatrix}
-1 & 2 \\
3 & 1
\end{bmatrix}
\begin{bmatrix}
6 & 5 \\
4 & 1
\end{bmatrix}
\]

38. Compute the product.

\[
\begin{bmatrix}
-1 & 2 \\
3 & 4 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 3
\end{bmatrix}
\]

39. Compute the product.

\[
\begin{bmatrix}
8 & -5 & 1 \\
-4 & 3 & -6 \\
4 & -3 & 9
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
40. Write the system of linear equations in matrix form (i.e., the $AX=B$ form).

\[
\begin{align*}
5x - 2y &= 9 \\
2x - 3y &= 8
\end{align*}
\]

41. The total output of loudspeaker systems of the Acrosomic Company in their three production facilities for May and June is given by the matrices $A$ and $B$, respectively, where

\[
A = \begin{bmatrix}
310 & 280 & 450 & 280 \\
480 & 370 & 580 & 0 \\
530 & 410 & 190 & 870
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
220 & 190 & 340 & 190 \\
390 & 300 & 440 & 30 \\
430 & 260 & 180 & 750
\end{bmatrix}
\]

The unit production costs and selling prices for these loudspeakers are given by matrices $C$ and $D$, respectively, where

\[
C = \begin{bmatrix}
110 \\
190 \\
260 \\
510
\end{bmatrix}
\quad \text{and} \quad
D = \begin{bmatrix}
160 \\
250 \\
350 \\
710
\end{bmatrix}
\]

Calculate $(A + B)(D - C)$. 

15
42. A farmer plans to plant two crops, A and B. The cost of cultivating crop A is $40/acre, whereas that of crop B is $60/acre. The farmer has a maximum of $7,400 available for land cultivation. Each acre of crop A requires 20 labor-hours, and each acre of crop B requires 25 labor-hours. The farmer has a maximum of 3,300 labor-hours available. If he expects to make a profit of $120/acre on crop A and $160/acre on crop B, how many acres of each crop should he plant in order to maximize his profit?

**NOTE:** I feel this is a great example of the type of question that can be on the take-home portion of the exam. If something like this were on the take-home, I would ask you to show ALL your work (i.e., the constraints, the feasible set drawing, etc).

_________ acres of crop A, _________ acres of crop B

What is the optimal profit? $________

43. Find the inverse of the given matrix, if it exists. If not, write *does not exist*. Verify your answer.

\[
\begin{bmatrix}
5 & 1 \\
9 & 2
\end{bmatrix}
\]

44. Find the inverse of the given matrix, if it exists. If not, write *does not exist*. Verify your answer.

\[
\begin{bmatrix}
1 & 1 \\
1 & 2
\end{bmatrix}
\]

Essay

45. Determine whether the statement is true or false. If it is true, explain why it is true. If it is false, give an example to show why it is false.

The point \((1, k)\) lies on the line with equation \(5x + 8y = 40\) if and only if \(k = \frac{35}{8}\).

Additional Problem

\[
\begin{bmatrix}
1 & 0 & 1 & -5 \\
0 & 1 & 3 & -1 \\
0 & -2 & -6 & 2
\end{bmatrix}
\]

Finish row-reducing this matrix & write solution.

If there are infinitely many solutions, write \(z = z\), \(x\) & \(y\) as functions of \(z\).

Don't forget to know the Ch. 3 content.
Some Math 104 Exam 1 Prep Questions

Answer Section

MULTIPLE CHOICE

1. ANS: B  PTS: 1
2. ANS: C  PTS: 1
3. ANS: A  PTS: 1
4. ANS: B  PTS: 1
5. ANS: A  PTS: 1
6. ANS: B  PTS: 1
7. ANS: A  PTS: 1
8. ANS: A  PTS: 1
9. ANS: C  PTS: 1
10. ANS: A  PTS: 1
11. ANS: A  PTS: 1
12. ANS: D  PTS: 1
13. ANS: E  PTS: 1
14. ANS: B  PTS: 1
15. ANS: A  PTS: 1
16. ANS: A  PTS: 1
17. ANS: B  PTS: 1
18. ANS: D  PTS: 1

NUMERIC RESPONSE

19. ANS: 30  PTS: 1
20. ANS: 3,500  PTS: 1
21. ANS: 3  PTS: 1
22. ANS: -3  PTS: 1
23. ANS: 100  PTS: 1
24. ANS: 272.5  PTS: 1
MATCHING

25. ANS: C  
   PTS: 1  
26. ANS: B  
   PTS: 1  
27. ANS: A  
   PTS: 1  

SHORT ANSWER

28. ANS:  
   no  
   PTS: 1  
29. ANS:  
   300; 90  
   PTS: 1  
30. ANS:  
   \[
   \begin{bmatrix}
   6 & 3 & 0 & 0 \\
   1 & -1 & 3 & 6 \\
   0 & 3 & -6 & 3 
   \end{bmatrix}
   \]  
   PTS: 1  
31. ANS:  
   yes  
   PTS: 1  
32. ANS:  
   \((-5, -1, 6)\)  
   PTS: 1  
33. ANS:  
   yes; 8; -4; 3 - z; z  
   PTS: 1  
34. ANS:  
   4; 3; 1; 5; 4; 1; 4; 4  
   PTS: 1  
35. ANS:  
   -6; -11; 10; 7  
   PTS: 1
36. ANS:
\[
\begin{bmatrix}
1 \\
-10
\end{bmatrix}
\]
PTS: 1

37. ANS:
\[
\begin{bmatrix}
2 & -3 \\
22 & 16
\end{bmatrix}
\]
PTS: 1

38. ANS:
\[
\begin{bmatrix}
7 & 1 & 5 \\
19 & 7 & 15 \\
4 & 1 & 3
\end{bmatrix}
\]
PTS: 1

39. ANS:
\[
\begin{bmatrix}
8 & -5 & 1 \\
-4 & 3 & -6 \\
4 & -3 & 9
\end{bmatrix}
\]
PTS: 1

40. ANS:
\[
\begin{bmatrix}
5 & -2 \\
2 & -3
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= \begin{bmatrix}
9 \\
8
\end{bmatrix}
\]
PTS: 1

41. ANS:
\[
\begin{bmatrix}
219,800 \\
181,500 \\
445,500
\end{bmatrix}
\]
PTS: 1

42. ANS:
65; 80; 20,600
PTS: 1
43. ANS:
\[
\begin{bmatrix}
2 & -1 \\
-9 & 5
\end{bmatrix}
\]

PTS: 1

44. ANS:
\[
\begin{bmatrix}
2 & -1 \\
-1 & 1
\end{bmatrix}
\]

PTS: 1

ESSAY

45. ANS:
True. If we substitute 1 instead of \(x\) and \(k\) instead of \(y\) and solve the equation for \(k\), we will find that \(k\) is equal to \(k = \frac{35}{8}\).

PTS: 1
Math 104 - Exam 2 Prep - Ch. 5, 6, and 10.1 (only)

Multiple Choice
Identify the choice that best completes the statement or answers the question.

1. A survey of 1,000 subscribers to the Los Angeles Times revealed that 700 people subscribe to the daily morning edition and 400 subscribe to both the daily and the Sunday editions.
   How many subscribe to the Sunday edition only?
   a. 800 people
   b. 1,000 people
   c. 200 people
   d. 700 people
   e. 300 people
   f. 500 people

2. In a card game, a 2-card hand consisting of an ace and either a face card or a 10 is called a “win”. If a standard 52-card deck is used, determine how many winning hands can be dealt. (A “face card” is a jack, queen, or king.)
   a. 64
   b. 20
   c. 40
   d. 48
   e. 16

3. A Social Security number has seven digits. How many Social Security numbers are possible?
   a. 6,000,000
   b. 720
   c. 5040
   d. 9,999,999
   e. 10,000,000

4. In a survey conducted by a union, members were asked to rate the importance of the following issues: (1) job security, (2) increased fringe benefits, and (3) improved working conditions. Three different responses were allowed for each issue. Among completed surveys, how many different responses to this survey were possible?
   a. 27
   b. 9
   c. 51
   d. 16
   e. 22
5. A computer dating service uses the results of its compatibility survey for arranging dates. The survey consists of 55 questions, each having four possible answers. How many different responses are possible if every question is answered?
   a. $4^{55}$
   b. $50^3$
   c. $3^{50}$
   d. $4^{50}$
   e. $55^4$

6. How many six-letter permutations can be formed from the first six letters of the alphabet?
   a. $C(6, 6) = 725$
   b. $C(6, 6) = 755$
   c. $P(6, 6) = 715$
   d. $P(6, 6) = 720$

7. Evaluate the expression.

   $P(m, 1)$
   a. $m(m - 1)$
   b. $m - 1$
   c. $m(m - 1) \over 2$
   d. $m$

8. A 2-member executive committee is to be formed from a 9-member board of directors. In how many ways can it be formed?
   a. 2
   b. 36
   c. 41
   d. 72
   e. 18

9. A quorum (minimum) of 11 voting members is required at all meetings of some association. If there is a total of 17 voting members in the group, find the number of ways this quorum can be formed. (NOTE: “minimum” means that number or more can make a quorum).
   a. 21,771
   b. 21,778
   c. 21,790
   d. 21,781
   e. 21,763
10. Let $S = \{a, b, c, d, e, f\}$ be a sample space of an experiment and let

$E = \{a, b\}$, $F = \{a, d, f\}$ and $G = \{b, c, e0\}$ be events of this experiment.

Are the events $F$ and $G$ mutually exclusive?

a. no
b. yes

11. A green and a red 6-sided die are rolled. What is the probability that the sum of the numbers shown uppermost is less than 7?

a. The probability is $\frac{5}{18}$
b. The probability is $\frac{5}{12}$
c. The probability is $\frac{1}{6}$
d. The probability is $\frac{3}{12}$

12. An experiment consists of selecting a card at random from a 52-card deck. Find the probability of the event that a heart or a jack is drawn.

a. $\frac{11}{52}$
b. $\frac{4}{13}$
c. $\frac{29}{52}$
d. $\frac{1}{2}$

13. An unbiased coin is tossed six times. Find the probability of the given event.

The coin lands heads more than once.

a. $\frac{57}{64}$
b. $\frac{1}{64}$
c. $\frac{3}{32}$
d. $\frac{63}{64}$
e. 1
f. 0
14. Ten people are selected at random. What is the probability that none of the people in this group have the same birthday?
   a. 0.486
   b. 0.331
   c. 0.88
   d. 0.139
   e. 0.793
   f. 0.641

15. The City Housing Authority has received 60 applications from qualified applicants for eight low-income apartments. Two of the apartments are on the north side of town, and six are on the south side. If the apartments are to be assigned by means of a lottery, what is the probability that a specific qualified applicant will be selected for one of these apartments? Round answer to two decimal places.
   a. 0.05
   b. 0.2
   c. 0.13
   d. 0.08
   e. 0.06

16. \( \{a, c\} \in \{a, b, c\} \)
   a. false
   b. true

17. Let \( A = \{2, 3, 4, 5\} \)
   Which of the following sets are equal to \( A? \)
   a. \( \{3, 4, 2, 5\} \)
   b. \( \{5, 2, 4, 3\} \)
   c. \( \{1, 2, 3, 4, 5\} \)

   a. a, b, and c
   b. a
   c. b
   d. a and b
   e. a and c

18. List all subsets of the set \( \{1, 6, 3\} \).
   a. \( \emptyset, \{1\}, \{6\}, \{3\}, \{1, 6\}, \{1, 3\}, \{6, 3\}, \{1, 6, 3\} \)
   b. \( \emptyset, \{1\}, \{6\}, \{3\}, \{1, 6\}, \{1, 3\}, \{6, 3\} \)
   c. \( \{1, 6, 3\}, \{1, 3, 6\}, \{6, 3, 1\}, \{6, 1, 3\}, \{3, 1, 6\}, \{3, 6, 1\} \)
   d. \( \{1\}, \{6\}, \{3\}, \{1, 6\}, \{1, 3\}, \{6, 3\} \)
19. Shade the portion of the accompanying figure that represents the set. The superscript ‘c’ means it is the complement of the set.

\[(A \cup B)^c \cap C\]

![Diagrams](image)

20. Let \( U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\} \) and \( A = \{5, 6, 7, 8, 9\} \). Find the set \( A^c \) where the superscript ‘c’ means it is the complement of set \( A \).

a. \( \{1, 2, 4, 6, 8, 9\} \)
b. \( \{3, 5, 7\} \)
c. \( \{2, 4, 6, 8, 9\} \)
d. \( \{1, 2, 3, 4\} \)
21. Let $U$ denote the set of all employees in a hospital. Let

\[ N = \{ x \in U \mid x \text{ is a nurse} \} \]
\[ D = \{ x \in U \mid x \text{ is a doctor} \} \]
\[ A = \{ x \in U \mid x \text{ is an administrator} \} \]
\[ M = \{ x \in U \mid x \text{ is a male} \} \]
\[ F = \{ x \in U \mid x \text{ is a female} \} \]

Describe the set in words.

\[ D \cap M^c \]
where the superscript ‘c’ means it is the complement of set.

a. The set of all employees in a hospital who are male doctors.
b. The set of all employees in a hospital who are male administrators.
c. The set of all employees in a hospital who are female doctors.
d. The set of all employees in a hospital who are female administrators.
e. The set of all employees in a hospital who are both doctors and administrators.

22. Let $U$ denote the set of all senators in Congress. Let

\[ D = \{ x \in U \mid x \text{ is a Democrat} \} \]
\[ R = \{ x \in U \mid x \text{ is a Republican} \} \]
\[ F = \{ x \in U \mid x \text{ is a female} \} \]
\[ L = \{ x \in U \mid x \text{ is a lawyer} \} \]

Write the set that represents the following statement:

The set of all Republicans who are female and are lawyers.

a. $R \cup (F \cup L)$
b. $D \cup (F \cap L)$
c. $D \cap (F \cup L)$
d. $R \cap (F \cap L)$
23. Refer to the accompanying figure and find the points that belong to the set \((B \cap C) \cap A^c\) where the superscript ‘c’ means it is the complement of set A.

\[ U \]

\[ B \quad C \]

\[ A \]

- a. \(s, w\)
- b. \(w\)
- c. \(w, r\)
- d. \(u\)

24. Find the future amount \(F\) if the principal \(P = 11,000\) is invested at the interest rate of 5% per year for 5.5 years, compounded quarterly.

- a. The accumulated amount is $14,585.32.
- b. The accumulated amount is $13,785.93.
- c. The accumulated amount is $14,100.05.
- d. The accumulated amount is $14,457.17.

25. If the future amount is $3,720 at the end of 3 years and the simple rate of interest is 8%/year, what is the principal?

- a. The principal is $3,500.
- b. The principal is $3,360.
- c. The principal is $3,000.
- d. The principal is $3,200.

**Numeric Response**

26. If \(n(A) = 17\), \(n(B) = 17\), \(n(C) = 12\), \(n(A \cap B) = 5\), \(n(A \cap C) = 5\), \(n(B \cap C) = 7\), and \(n(A \cup B \cup C) = 30\), find \(n(A \cap B \cap C)\).

\[ n(A \cap B \cap C) = \__________ \]
27. Three different types of monthly commuter passes are offered by a city's local transit authority for each group of three different groups of passengers. How many different kinds of passes must be possible?
   
   ______ possible passes

28. Computers manufactured by a certain company have a serial number consisting of a letter of the alphabet followed by a four-digit number. If all the serial numbers of this type have been used, how many sets have already been manufactured?
   
   ______ sets

29. In how many ways can an investor select four mutual funds for his investment portfolio from a recommended list of nine mutual funds?
   
   ______ ways

30. Two cards are selected at random without replacement from a well-shuffled deck of 52 playing cards. Find the probability of the given event. Round your answer to the nearest thousandth, if necessary.

   Two black cards are drawn.

   The probability is ________.

---

Binomial Expansion

(39) What is the coefficient of \(x^2y^5\) in \((2x - 3y)^7\)?

(40) How many distinguishably different arrangements are there of the word "rearrangement"?
31. A list of poker hands ranked in order from the highest to the lowest is shown in the table, along with a description and example of each hand. Use the table to answer the problem.

<table>
<thead>
<tr>
<th>Hand</th>
<th>Description</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Straight</td>
<td>5 cards in sequence</td>
<td>A ♠ 2 ♠ 3 ♠ 4 ♠ 5 ♠</td>
</tr>
<tr>
<td>flush</td>
<td>in the same suit</td>
<td></td>
</tr>
<tr>
<td>Four of a kind</td>
<td>4 cards of the same rank and any other card</td>
<td>K ♠ K ♣ K ♣ 2 ♠</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Full house</td>
<td>3 of a kind and a pair</td>
<td>3 ♠ 3 ♠ 7 ♠ 7 ♠</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Flush</td>
<td>5 cards of the same suit that are not all in sequence</td>
<td>5 ♠ 6 ♠ 9 ♠ J ♠ K ♠</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Straight</td>
<td>5 cards in sequence but not all of the same suit</td>
<td>10 ♣ J ♣ Q ♣ K ♣ A ♣</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Three of a kind</td>
<td>3 cards of the same rank and 2 unmatched cards</td>
<td>K ♠ K ♣ K ♣ 2 ♣ 4 ♣</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Two pair</td>
<td>2 cards of the same rank and 2 cards of any other rank with an unmatched card</td>
<td>K ♠ K ♣ 2 ♣ 4 ♣</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>One pair</td>
<td>2 cards of the same rank and 3 unmatched cards</td>
<td>K ♠ K ♣ 5 ♣ 2 ♣ 4 ♣</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

If a 5-card poker hand is dealt from a well-shuffled deck of 52 cards, what is the probability of being dealt a straight? (an ace may be played as either a high or low card in a straight sequence)

Round your answer to five decimal places, if necessary.

The probability is __________.

32. A bank deposit paying simple interest grew from an initial sum of $1,000 to a sum of $1,025 in 3 mo. Find the interest rate.

\[ r = __________ \% / \text{year} \]

Short Answer

33. Let \( A = \{1, 2, 3, 4, 5\} \). Determine whether the given statement is true or false.

\[ 7 \in A \]
34. In a survey of 120 consumers conducted in a shopping mall, 81 consumers indicated that they buy brand A of a certain product, 68 buy brand B, and 43 buy both brands. How many consumers participating in the survey buy at least one of these brands? How many consumers participating in the survey buy exactly one of these brands? How many consumers participating in the survey buy only brand A? How many consumers participating in the survey buy none of these brands?

_________ consumers bought at least one of these brands

_________ consumers bought exactly one of these brands

_________ consumers bought only brand A

_________ consumers bought none of these brands

35. If S is our sample space, and \( S = \{1, 2, 3, 4, 5, 6\} \), \( E = \{2, 4, 6\} \), \( F = \{1, 3, 5\} \) and \( G = \{2, 3\} \).

Find the event \((E \cap F \cap G)^c\).

By the way, the superscript ‘c’ means it is the complement of set.

Express your answer using set notation.

36. Let \( S = \{s_1, s_2, s_3, s_4, s_5, s_6\} \) be the sample space associated with an experiment having the following probability distribution:

<table>
<thead>
<tr>
<th>Outcome</th>
<th>(s_1)</th>
<th>(s_2)</th>
<th>(s_3)</th>
<th>(s_4)</th>
<th>(s_5)</th>
<th>(s_6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>(\frac{1}{3})</td>
<td>(\frac{1}{18})</td>
<td>(\frac{1}{6})</td>
<td>(\frac{1}{36})</td>
<td>(\frac{1}{18})</td>
<td>(\frac{1}{3})</td>
</tr>
</tbody>
</table>

Find the probability of the event \(A = \{s_1, s_3\}\).

37. Four balls are selected at random without replacement from an urn containing three white balls and six blue balls. Find the probability of the given event. Give your answer in the form of a fraction.

Two or three of the balls are white.

38. A shelf in the Metro Department Store contains 70 colored ink cartridges for a popular ink-jet printer. Six of the cartridges are defective. A customer selects 2 of these cartridges at random from the shelf.

What is the probability that both are defective?

_________

What is the probability that at least 1 is defective?
Math 104 - Exam 2 Prep - Ch. 5, 6, and 10.1 (only)
Answer Section

MULTIPLE CHOICE

1. ANS: E  PTS:  1
2. ANS: A  PTS:  1
3. ANS: E  PTS:  1
4. ANS: A  PTS:  1
5. ANS: A  PTS:  1
6. ANS: D  PTS:  1
7. ANS: D  PTS:  1
8. ANS: B  PTS:  1
9. ANS: B  PTS:  1
10. ANS: B  PTS:  1
11. ANS: B  PTS:  1
12. ANS: B  PTS:  1
13. ANS: A  PTS:  1
14. ANS: C  PTS:  1
15. ANS: C  PTS:  1
16. ANS: A  PTS:  1
17. ANS: D  PTS:  1
18. ANS: A  PTS:  1
19. ANS: B  PTS:  1
20. ANS: D  PTS:  1
21. ANS: C  PTS:  1
22. ANS: D  PTS:  1
23. ANS: B  PTS:  1
24. ANS: D  PTS:  1
25. ANS: C  PTS:  1

NUMERIC RESPONSE

26. ANS: 1
   PTS:  1
27. ANS: 9
   PTS:  1
28. ANS: 260,000
   PTS:  1
29. ANS: 126
   PTS:  1
30. ANS: 0.245
     PTS: 1
31. ANS: 0.00392
     PTS: 1
32. ANS: 10
     PTS: 1

SHORT ANSWER

33. ANS:
    false
     PTS: 1
34. ANS:
    106; 63; 38; 14
     PTS: 1
35. ANS:
    \((E \cap F \cap G)^c = \{1,2,3,4,5,6\}\)
     PTS: 1
36. ANS:
    \(\frac{1}{2}\)
     PTS: 1
37. ANS:
    \(\frac{17}{42}\)
     PTS: 1
38. ANS:
    0.006; 0.165
     PTS: 1
Multiple Choice
Identify the choice that best completes the statement or answers the question.

1. Find the simple interest on a $400 investment made for 5 years at an interest rate of 7%/year. What is the accumulated amount?
   a. The simple interest is $140, the accumulated amount is $540.
   b. The simple interest is $115, the accumulated amount is $515.
   c. The simple interest is $120, the accumulated amount is $520.
   d. The simple interest is $125, the accumulated amount is $555.

2. If the accumulated amount is $3,720 at the end of 3 years and the simple rate of interest is 8%/year, what is the principal?
   a. The principal is $3,500.
   b. The principal is $3,360.
   c. The principal is $3,000.
   d. The principal is $3,200.

3. Find the accumulated amount $A$ if the principal $P = $11,000 is invested at the interest rate of $r = 5\%$ per year for $t = 5.5$ years, compounded quarterly.
   a. The accumulated amount is $14,585.32.
   b. The accumulated amount is $13,785.93.
   c. The accumulated amount is $14,100.05.
   d. The accumulated amount is $14,457.17.

4. Find the accumulated amount $A$ if the principal $P = $170,000 is invested at the interest rate of $r = 5\%$ per year for $t = \frac{1}{4}$ years compounded monthly.
   a. The accumulated amount is $231,540.40.
   b. The accumulated amount is $232,339.79.
   c. The accumulated amount is $232,211.64.
   d. The accumulated amount is $231,854.52.
5. Find the future value of an increasing annuity of 10 yearly payments of $1,800 that earn interest at 10% per year, compounded annually.
   a. $4,668.74
   b. $28,687.36
   c. $87,798.04
   d. $3,600.00

6. Lupe made a down payment of $1,500 toward the purchase of a new car. To pay the balance of the purchase price, she has secured a loan from her bank at the rate of 12%/year compounded monthly. Under the terms of her finance agreement, she is required to make payments of $240/month for 30 mo. What is the cash price of the car?

   Math 104 Students: Is this a decreasing OR increasing annuity problem?
   a. $9,848.37
   b. $6,193.85
   c. $3,433.24
   d. $7,693.85

7. Robin, who is self-employed, contributes $4,000/year into a Keogh account (i.e., a retirement account for self-employed people). How much will he have in the account after 15 years if the account earns interest at the rate of 6.5%/year compounded yearly?

   Math 104 Students: Is this a decreasing OR increasing annuity problem?
   a. $96,728.68
   b. $10,287.36
   c. $158,267.14
   d. $3,771.28

8. Find the present value of a decreasing annuity of $600 payments each made quarterly over 5 years and earning interest at 4% per year compounded quarterly.

   a. $8,154.20
   b. $2,671.09
   c. $10,827.33
   d. $56,916.87
9. Find the periodic payment $R$ required to amortize a loan of $P$ dollars over $t$ years with interest earned at the rate of $r\%$/year compounded $m$ times a year. Round your answer to the nearest cent.

**Math 104 Students:** This is an amortization problem, but note that your book formula has an “$n$”. You will need to know what that $n$-value means in this context. In our formula, $n$ was the total number of interest pay periods.

$P = 120,000, \ r = 7, \ t = 8, \ m = 2$

a. $R = 9,864.83$

b. $R = 10,279.66$

c. $R = 9,922.18$

d. $R = 9,798.65$

10. The price of a new car is $12,000. Assume an individual makes a down payment of 25% toward the purchase of the car and secures financing for the balance at the rate of 9%/year compounded monthly.

What monthly payment will she be required to make if the car is financed over a period of 42 months? What will the interest charges be if she elects the 42-month plan? Round your answers to the nearest cent.

a. $R = 274.13; \text{ interest charges} = 1,489.36$

b. $R = 250.60; \text{ interest charges} = 1,525.20$

c. $R = 250.60; \text{ interest charges} = 1,489.36$

d. $R = 274.13; \text{ interest charges} = 1,525.20$

**Short Answer**

11. In the last 2 years, mutual fund A grew at the rate of 10.7%/year compounded quarterly. Over the same period, mutual fund B grew at the rate of 10.8%/year compounded semi-annually. Which mutual fund has a better rate of return? (Answer A or B.)

And provide a detailed explanation of your answer for FULL CREDIT!

The next three pages have Questions on Ch. 11 (difference equations) that appear in WeBWorK.
Math 104 Students: Welcome to the fun world of sequences!!! All of our sequences will begin with the 0th term, i.e., something usually of the form \(a_0\) or \(y_0\). In this and the next problem you will be asked to give what I called a **closed form** in class. For example, the sequence 1, 2, 3, ..., has closed form \(y_n = n + 1\). Another example: the sequence \(-1, 1, -1, 1, \ldots\) has closed form \(y_n = (-1)^{n+1}\).

For the sequence 1, 4, 7, 10, ..., a closed form is \(y_n = \frac{2n - 1}{2}\).

**ATTENTION:** Make sure your answer works. How do you check it? Try testing some values of \(n\). For example, if you answered something like \(y_n = \frac{4n + 1}{n+1}\), then make sure that setting \(n = 0\) tells you that \(y_0 = 1\). And yay, that works since \(\frac{4(0) + 1}{0+1} = 1\), but does setting \(n = 1\) tell you that \(y_1 = 4\)? No, it doesn’t. So test your answer on a bunch of \(n\)-values.

**Hint:** To enter an exponent into WeBWorK, you would use the carat symbol \(^{\text{\textasciicircum}}\) (above the 6-key on your keyboard), or use the ** symbol. For example, to write \(7^{n-4}\), you would enter either \(7^{n-4}\) or \(7**(n-4)\).

3. (3 pts) local/Library/Rochester/setAlgebra36SeqSeries/arw10.1.7.pg

For the sequence \(a_n = 12 + (-1)^n\),

| its first term (i.e., \(a_0\)) is | ____ |
| its second term (i.e., \(a_1\)) is | ____ |
| its third term (i.e., \(a_2\)) is | ____ |
| its fourth term (i.e., \(a_3\)) is | ____ |
| its 100th term (i.e., \(a_{100}\)) is | ____ |

Math 104 Students: The following is an example of a recursion formula.

For the sequence given by the recursion formula \(a_n = 2(a_{n-1} - 2)\) with initial condition \(a_0 = 1\),

its first term (i.e., \(a_0\)) is ____;
its second term is ____;
its third term is ____;
its fourth term is ____;
its fifth term is ____.

5. (3 pts) local/Library/Rochester/setAlgebra36SeqSeries/arw10.1.15.pg

Math 104 Students: The following is a recursion formula problem. However unlike the previous problem, the problem below has **TWO** initial conditions. So it is like the Fibonacci example which I did in class.

For the sequence given by the recursion formula \(a_n = a_{n-1} + a_{n-2}\) with initial conditions \(a_0 = 3, a_1 = 4\),

| its first term (i.e., \(a_0\)) is | ____ |
| its second term (i.e., \(a_1\)) is | ____ |
| its third term is | ____ |
| its fourth term is | ____ |
| its fifth term is | ____ |

6. (3 pts) local/Library/Rochester/setAlgebra36SeqSeries/arw10.1.17.pg

Consider the sequence given by the closed form \(a_n = 5n + 3\).

| Its 9th term is | ____ |
| Its 10th term is | ____ |

7. (3 pts) local/Library/Rochester/setAlgebra36SeqSeries/recursive1.pg

Math 104 Students: This last problem is asking you think critically. I advise you to FIRST use the recursion formula on \(a_4\) (i.e., the fact that \(a_4 = 3a_3 + 6a_2 - 3a_1\)) to find the value of \(a_3\). Then, proceed to find \(a_2\) using a similar method, and then finally find \(a_1\).

Suppose \(a_6 = 3a_5 + 6a_4 - 3a_3\) and \(a_4 = -6, a_5 = -33,\) and \(a_6 = -129\). Find \(a_1, a_2,\) and \(a_3\).

\[a_1 = \quad ____ \]
\[a_2 = \quad ____ \]
\[a_3 = \quad ____ \]
Suppose that you deposit $503 into a bank charging 7% simple interest.

a) Let \( y_n \) equal the amount at the end of \( n \) years.

Our initial deposit is \( y_0 = \) __________.

Then the difference equation that models this simple interest problem is the following:

\[ y_n = y_{n-1} + \]

b) Find an explicit formula for \( y_n \). And write it below here (note that the "\( y_n = \)" part is already written for you).

\( y_n = \) __________

c) The amount of money in the bank after 5 years is:

_______

5. (3 pts) local/aba-ch11-differenceEQ-6.png

Re-read Example 3 on pg.534-535 in the book before starting this problem. Then solve problem 6 on pg.537, putting your answers below.

a) Let \( y_n \) equal the amount at the end of \( n \) quarter-years (since interest is compounded quarterly in this problem 6 from the book).

Our initial deposit is \( y_0 = \) __________.

Then the difference equation that models this compound interest problem is the following:

\[ y_n = y_{n-1} - \]

b) Find an explicit formula for \( y_n \). And write it below here (note that the "\( y_n = \)" part is already written for you).

\( y_n = \) __________

c) The amount of money in the bank after 5 years is:

_______

GOOD ADVICE: This is like the previous problem -- if it helps, then review Example 2 on pg.534.

4. (3 pts) local/aba-ch11-differenceEQ-5.png

GOOD ADVICE: This is like the previous problem -- if it helps, then review that Example 2 on pg.534.
REVIEW: Finance, Difference Equations, and Logic

Answer Section

MULTIPLE CHOICE

1. ANS: A  PTS: 1
2. ANS: C  PTS: 1
3. ANS: D  PTS: 1
4. ANS: C  PTS: 1
5. ANS: B  PTS: 1
6. ANS: D  PTS: 1
7. ANS: A  PTS: 1
8. ANS: C  PTS: 1
9. ANS: C  PTS: 1
10. ANS: B  PTS: 1

SHORT ANSWER

11. ANS:
   A
   PTS: 1

---

Ch. 11 - Difference Eqns:

1. $y_n = 6, y_1 = 15, y_2 = 42, y_3 = 123;
   y_n = 366, a = 3, b = -3$
   and $\frac{b}{1-a} = 1.5$

2. $y_n = 40 + 43 (1.05)^n$
   $y_{17} = 58, 55, 679$

3. $80, 32, 80 + 3.2n, 96$

4. $503, 35, 21, 503 + (35, 2) n$
   $\$679.05$

5. $80, 1.02, 80 \cdot (1.02)^n$
   $\$118.88$

6. $561, 1.005, 561 \cdot (1.005)^n$
   $\$852.93$
Here are some suggested problems:

<table>
<thead>
<tr>
<th>chapter</th>
<th>page</th>
<th>question</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>15</td>
<td>28,32</td>
</tr>
<tr>
<td></td>
<td>21</td>
<td>3,11</td>
</tr>
<tr>
<td></td>
<td>31</td>
<td>47,48</td>
</tr>
<tr>
<td></td>
<td>46</td>
<td>28</td>
</tr>
<tr>
<td>2</td>
<td>79</td>
<td>43,47,55</td>
</tr>
<tr>
<td>3</td>
<td>114</td>
<td>14</td>
</tr>
<tr>
<td></td>
<td>122</td>
<td>5,6,7,8</td>
</tr>
<tr>
<td></td>
<td>123</td>
<td>33</td>
</tr>
<tr>
<td>5</td>
<td>205</td>
<td>39--44</td>
</tr>
<tr>
<td></td>
<td>211</td>
<td>37--42</td>
</tr>
<tr>
<td></td>
<td>218</td>
<td>27,37,57</td>
</tr>
<tr>
<td></td>
<td>224</td>
<td>21,22,33,35</td>
</tr>
<tr>
<td></td>
<td>229</td>
<td>17,18,40,44,47</td>
</tr>
<tr>
<td>6</td>
<td>249</td>
<td>22</td>
</tr>
<tr>
<td></td>
<td>258</td>
<td>13,14</td>
</tr>
<tr>
<td></td>
<td>266</td>
<td>9</td>
</tr>
<tr>
<td></td>
<td>267</td>
<td>42</td>
</tr>
<tr>
<td>10</td>
<td>437</td>
<td>15,17</td>
</tr>
<tr>
<td></td>
<td>446</td>
<td>2,6,7</td>
</tr>
<tr>
<td></td>
<td>455</td>
<td>5,7,8,9,10,18</td>
</tr>
<tr>
<td>11</td>
<td>484</td>
<td>5,6,14</td>
</tr>
<tr>
<td></td>
<td>485</td>
<td>26</td>
</tr>
<tr>
<td></td>
<td>491</td>
<td>21,22</td>
</tr>
</tbody>
</table>
REMEMBER!! Don’t peek at these answers until you have done the problem!!

Pg.15 #28) line with intercepts (0,6) and (2,0) and shade BELOW this line since feasible set is ABOVE this line
Pg.15 #32) line with intercepts (0,-2) and (2,0) and shade ABOVE this line since feasible set is BELOW this line
Pg.21 #3) (2,1)
Pg.21 #11) x=-7/9 and y=-22/9
Pg.31 #47) y = -x+2
Pg.31 #48) y = (1/2)x
Pg.46 #28) (a) is graph (C), (b) is graph (A), (c) is graph (B), and (d) is graph (D).
Pg.79 #43) \( \begin{cases} 2x + 3y = 6 \\ 4x + 5y = 7 \end{cases} \)
Pg.79 #47) \[ \begin{bmatrix} 3 \\ 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \]
Pg.79 #55) (a) \[ \begin{bmatrix} 340 \\ 265 \end{bmatrix} \], (b) Mike’s clothes cost $340 and Don’s clothes cost $265, (c) \[ \begin{bmatrix} 25 \\ 18.75 \\ 62.50 \end{bmatrix} \], and (d) the costs of the three items of clothing after a 25% increase.

Pg.114 #14) The answer to (a) is the table below:

<table>
<thead>
<tr>
<th></th>
<th>Denim</th>
<th>Hooded Fleece</th>
<th>Available</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cutting</td>
<td>2 hours</td>
<td>1 hour</td>
<td>42 hours</td>
</tr>
<tr>
<td>Sewing</td>
<td>2 hours</td>
<td>4 hours</td>
<td>90 hours</td>
</tr>
<tr>
<td>Finishing</td>
<td>1 hour</td>
<td>1 hour</td>
<td>27 hours</td>
</tr>
<tr>
<td>Profit</td>
<td>$9</td>
<td>$5</td>
<td></td>
</tr>
</tbody>
</table>

(b) \( 2x + y \leq 42, 2x + 4y \leq 90, x + y \leq 27 \)
(c) \( x \geq 0, y \geq 0 \)
(d) \( 9x + 5y \)
(e) Use Desmos to verify your graph calculation.

Pg.122 #5) (0,5)
Pg.122 #6) (3,3)
Pg.122 #7) (3,3)
Pg.122 #8) (0,0)
Pg.123 #33) Produce 9 hockey games and 8 soccer games each day.
Pg.205 #39) 190
Pg.205 #40) 80
Pg.205 #41) 180
Pg.205 #42) 160
Pg.205 #43) 210
Pg.205 #44) 50
Pg.211 #37) \( 2^6 = 64 \)
Pg.211 #38) \( 2^5 = 32 \)
Pg.211 #39) $2^5 = 32$
Pg.211 #40) $3^5 = 243$
Pg.211 #41) $4^{10} = 1,048,576$
Pg.211 #42) $5^{10} = 9,765,625$
Pg.218 #27) $4!$
Pg.218 #37) $C(100,3)=161,700; C(7,3)=35$
Pg.218 #57) $4!*P(4,3)*P(5,3)*P(6,3)*P(7,3) = 870,912,000$
Pg.224 #21) (a) $C(12,4)=495$, (b) $C(8,4)=70$, (c) $C(8,2)*C(4,2)=168$, (d) $C(8,3)*4 + 70 = 294$
Pg.224 #22) (a) $C(15,6)=5005$, (b) $C(9,6)=84$, (c) $C(6,2)*C(9,4)=1890$, and (d) $5005 – (84 + 6*C(9,5)) = 4165$
Pg.224 #33) $C(10,5)*P(21,5) = 615,353,760$
Pg.229 #17) 64
Pg.229 #18) 128
Pg.228 #30) 16 since there are $C(4,0)+C(4,1)+C(4,2)+C(4,3)+C(4,4)$ options
Pg.229 #44) $2^6-1 = 63$. Note that selecting ZERO desserts is an option.
Pg.229 #47) $2^7-C(7,6)-C(7,7) = 120$. Note that selecting ZERO appetizers is an option.
Pg.249 #22) $\{2,3,4,5,6,7,8,9,10,11,12\}$
Pg.258 #13) (a) $5/36$ which is approximately .1389, (b) $1/6$ which is approximately .1667
Pg.258 #14) (a) $\frac{3}{5}$, (b) $\frac{4}{5}$ = .75
Pg.266 #9) Approximately .8333
Pg.267 #42) $6/20 = .3$
Pg.437 #15) $\$6505.63; \$505.63$
Pg.437 #17) $\$12,824.32$
Pg.446 #2) $i=.02$, $n=20$, $R=\$8231.34$, and $F=\$200,000$
Pg.446 #6) $\$31,798.79$
Pg.446 #7) $\$139$
Pg.455 #5) $\$2270$
Pg.455 #7) $\$608.13$
Pg.455 #8) $\$1311.66$
Pg.455 #9) $\$143,000$
Pg.455 #10) $\$208,000$
Pg.455 #18) (a) $\$1988.29$, (b) $\$666,487$, (c) $\$296,487$, (d) $\$44,724.98$, (e) $\$23,064.84$, (f) $\$2199.34$.
Pg.484 #5) $a=-2/3$, $b=15$; $b/(1-a) = 9$
Pg.484 #6) $a=.5$, $b=-4$; $b/(1-a) = -8$
Pg.484 #14) (a) -2, -1, 1, 5, 13, (b) ignore, and (c) $y_n = -3 + 2^n$
Pg.485 #26) (a) $y_n = .85y_{n-1}$ with $y_0 = 20000$, (b) $y_n = 20000(.85)^n$, (c) $\$8,874.11$
Pg.491 #21) $y_n = y_{n-1} - 2000$ with $y_0 = 50000$ and it has closed form $y_n = 50000 - 2000n$
Pg.491 #22) $y_n = \frac{23}{25} \cdot y_{n-1}$ with $y_0 = 50000$ and it has closed form $y_n = 50000 \cdot \left(\frac{23}{25}\right)^n$
If Covid-19 lessens its control of humanity by the end of the semester, there be a math reunion in the Spring 2021 semester. Every semester except for this past semester, I have had a Math 104 reunion at my home. Let’s hope that we can gather together in a social setting at least by next semester and every semester after that 😊! Below is a pic of a recent reunion gathering: