Towards a Schur-Weyl Duality for the Alternating Group

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Abstract

Using known branching rules of the symmetric group, $S_n$, and applications of Clifford theory to its index 2 subgroup, the alternating group $A_n$, we deduce the branching rules for $A_n$. This is a key first step in coming up with a version of Schur-Weyl duality for this group. The centralizer of the action of $A_n$ on $k$-fold tensor copies of its permutation representation should be a super-algebra of the partition algebra.

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1 Introduction

1.1 HISTORY: Schur-Weyl Duality from 1927 to the present

In 1927, Schur proved his classical result on the double centralizer theory of the symmetric group, \( S_k \), and the general linear group, \( GL_n(\mathbb{C}) \). Let \( V = \mathbb{C}^n \) be a vector space over \( \mathbb{C} \). \( GL_n(\mathbb{C}) \) acts naturally on \( V \) by left multiplication. Now, consider \( k \)-fold tensor copies of \( V \), we denote as \( V^\otimes k \). We may extend the action of \( GL_n(\mathbb{C}) \) on \( V^\otimes k \) by \( g \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_k) = g \cdot v_1 \otimes g \cdot v_2 \otimes \cdots \otimes g \cdot v_k \). Also, \( S_k \) acts naturally on \( V^\otimes k \) by permuting the tensor places, that is, \( \sigma \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_k) = v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(k)} \). Schur’s famed result said that the image of the group algebra of \( S_k \) is the full centralizer of the action of \( GL_n(\mathbb{C}) \). And conversely, the image of the group algebra of \( GL_n(\mathbb{C}) \) is the full centralizer of the action of \( S_k \). That is, if we name the representations thusly,

\[
\mathbb{C} \otimes GL_n(\mathbb{C}) \xrightarrow{\phi} \text{End}(V^\otimes k) \quad \xleftarrow{\psi} \quad \mathbb{C} S_k
\]

Then, we have \( \phi(\mathbb{C} GL_n(\mathbb{C})) = \text{End}_{S_k}(V^\otimes k) \) and \( \psi(\mathbb{C} S_k) = \text{End}_{GL_n(\mathbb{C})}(V^\otimes k) \).

In 1937, Richard Brauer went on to look at interesting subgroups of \( GL_n(\mathbb{C}) \), namely the orthogonal group \( O_n(\mathbb{C}) \) and the symplectic group \( Sp_n(\mathbb{C}) \). He discovered that the centralizer of the action of these subgroups on \( V^\otimes k \) corresponds to a set of diagrams on \( 2k \) vertices and \( k \) edges such that there are two rows of \( k \) vertices each, and every vertex has degree one so that the vertices are paired up by the edges. This algebra of diagrams was subsequently called the Brauer algebra. The symmetric group \( S_k \) imbeds in this Brauer algebra as a subset of permutation diagrams – that is, all \( k \) edges travel from the top row to the bottom row. Surprisingly, it was six decades later that the centralizer of the special orthogonal group, \( SO_n \), was studied in detail. This work was completed by Cheryl Grood in her thesis in 1998 under the supervision of Georgia Benkart.

There are other interesting subgroups of \( GL_n(\mathbb{C}) \) that also have recently been investigated. We may view \( S_n \) as sitting inside \( GL_n(\mathbb{C}) \) as the set of permutation matrices so that \( V \) is the permutation representation. Then restricting the action of \( GL_n(\mathbb{C}) \) to this subgroup, we get an action of \( S_n \) on \( V^\otimes k \). Simultaneously, Vaughn Jones and also Paul Martin investigated the centralizer of this action in 1993. They deduced that this centralizer corresponds to a set of diagrams which we call the partition algebra. This partition algebra, like the Brauer algebra, is comprised of diagrams with \( 2k \) vertices arranged in two rows of \( k \) vertices each; however, vertices are now allowed to have degree zero, and hence we may have less than \( k \).
edges.

In the spring of 2008, while at MSRI, I had many discussions with Tom Halverson about investigating the centralizer of $\mathfrak{A}_n$ viewed as sitting inside $\text{GL}_n(\mathbb{C})$ as determinant 1 permutation matrices. The centralizer of this subgroup acting on $V^\otimes k$ should be a super-algebra of the partition algebra. In the following pages, I will discuss my results towards this end.

I would like to thank Georgia Benkart and Steve Doty for many fruitful discussions on centralizer algebras and Schur-Weyl duality in general, Vaughn Jones for speaking with me about his partition algebra, and my advisor Fred Goodman for supervising this project and helping me clean-up all the proofs. Lastly, I thank Tom Halverson for suggesting this project.

2 Branching rules for the alternating group

2.1 Branching rules for $\mathfrak{S}_n$

Theorem 1 ($\mathfrak{S}_n$ Branching Rules). Let $S^{(n-1)}_\mu$ be an irreducible $\mathfrak{S}_{n-1}$-module and $S^{(n)}_\lambda$ be an irreducible $\mathfrak{S}_n$-module. Then we have the following going up and going down branching rules.

\[
\begin{align*}
\text{Ind}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} S^{(n-1)}_\mu &\cong \bigoplus_{\lambda=\mu+\square} S^{(n)}_\lambda \\
\text{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} S^{(n)}_\lambda &\cong \bigoplus_{\mu=\lambda-\square} S^{(n-1)}_\mu 
\end{align*}
\]

where the notation $\lambda = \mu + \square$ (respectively, $\mu = \lambda - \square$) means to sum over all the ways of adding one box to $\mu$ (respectively, removing one box from $\lambda$).

Proof. This is a well-known result and follows, for example, from by James [3, pg ?].

2.2 Clifford theory for subgroups of prime index

Global assumption for the next two lemmas: Let $H \triangleleft G$ with $[G : H] = p$ for some prime $p$. Some of the ideas in this subsection are briefly sketched in [1] by Bröcker and tomDieck. However, in the following, we provide full proofs for many of their missing details.

Lemma 2. Let $U$ be a $\mathbb{C}[H]$-module. Then, we have

\[
\text{Res}_H^G \text{Ind}_H^G U \cong \bigoplus_{x \in G/H} U_x
\]
where the $U_x$ are twisted $H$-modules with action $h \cdot u = xhx^{-1}u$.

**Proof.** Let $U$ be a complex $H$-module. Then $\text{Ind}^G_H U = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} U$ by definition of induction. Since $G$ is a disjoint union of left cosets, we have the following implication:

$$G = \bigsqcup_{1 \leq i \leq p} g_iH \implies \mathbb{C}[G] \cong \bigoplus_{g_iH \in G/H} \mathbb{C}[g_iH]$$

Warning! This is not true for Lie groups in general, but in our application of $G = \mathfrak{S}_n$ and $H = \mathfrak{A}_n$ we are fine. Thus, we have

$$\text{Ind}^G_H U = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} U \cong \bigoplus_{i=1}^p \mathbb{C}[g_iH] \otimes_{\mathbb{C}[H]} U \text{ isomorphic as } \mathbb{C}\text{-vector spaces}$$

Thus, we get our desired result. \hfill \square

**Lemma 3.** Let $V$ be a $\mathbb{C}[G]$-module. Then, we have

$$\text{Ind}^G_H \text{Res}^H_G V \cong \bigoplus_{k \in \mathbb{Z}/p} V \otimes_G \Omega(k)$$

where $\Omega(k)$ as $k = 0, 1, \ldots, p-1$ are the $p$ distinct irreducible representations of $G/H \cong \mathbb{Z}/p\mathbb{Z}$ given by $\Omega(k) : G/H \rightarrow \text{GL}_n(\mathbb{C})$ via $x \mapsto (z \mapsto e^{\frac{2\pi ik}{p}} z)$ where $x$ is a fixed generator of $G/H$.

**Proof.** Let $V$ be a complex $G$-module. The map $G \xrightarrow{\pi} G/H \xrightarrow{\Omega(k)} \text{GL}_n(\mathbb{C})$ makes $\Omega(k)$ into a $G$-module. Also, $\Omega(k) \otimes \Omega(l) = \Omega(k+l)$ and $\Omega(0) = \mathbb{C}$ as the trivial representation. Thus, $\mathbb{Z}/p\mathbb{Z}$ acts on the $G$-modules $V \otimes \Omega(k)$. Therefore,

$$\{V \cong V \otimes \Omega(0), V \otimes \Omega(1), \ldots, V \otimes \Omega(p-1)\}$$

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are all isomorphic or all mutually non-isomorphic $G$-modules. We get the following string of equalities:

\[
\begin{align*}
\text{Ind}_H^G \text{Res}_H^G V &= \text{Ind}_H^G (\text{Res}_H^G V \otimes 1_H) \\
&\cong V \otimes \text{Ind}_H^G 1_H \quad \text{see lemmette below} \\
&= V \otimes \mathbb{C}[G/H] \\
&= V \otimes \sum_{k=0}^{p-1} \Omega(k) \\
&= \sum_{k=0}^{p-1} (V \otimes \Omega(k))
\end{align*}
\]

Thus, we get our desired result.

\[\square\]

**Lemmette 4.** $\text{Ind}_H^G (\text{Res}_H^G (M) \otimes_H N) \cong M \otimes_G \text{Ind}(N)$ where $M$ is a $G$-module and $N$ is an $H$-module.

*Proof.* We leave this tedious proof out. But in our case of $G = \mathfrak{S}_n$ and $H = \mathfrak{S}_{n-1}$, the isomorphism is given by the fact that $\text{Ind}_H^G (W) \cong \mathbb{C}[G] \otimes_H W$ where $W$ is an $H$-module and the map $\mathbb{C}[G] \otimes_H (\text{Res}(M) \otimes_C N) \rightarrow M \otimes_G \text{Ind}(N)$ by $\sigma \otimes (m \otimes n) \mapsto \sigma m \otimes (\sigma \otimes n)$. Note: Later we will use $G = \mathfrak{A}_n$ and $H = \mathfrak{A}_{n-1}$. The same omitted proof works in that case too.

\[\square\]

We now prove a main theorem of this section. First, we define two types of irreducible $G$ and $H$ modules in the following table. We will let $\hat{G}$ and $\hat{H}$ denote the set of irreducible modules for $G$ and $H$ respectively.

\[
\begin{align*}
V \in \hat{G} \text{ is type I} & \iff \text{ all } V \otimes \Omega(k) \text{ distinct for } k = 0, 1, \ldots, p-1 \\
V \in \hat{G} \text{ is type II} & \iff \text{ all } V \otimes \Omega(k) \text{ isomorphic for } k = 0, 1, \ldots, p-1 \\
U \in \hat{H} \text{ is type I} & \iff \text{ all } U_x \text{ isomorphic for } x \in G/H \\
U \in \hat{H} \text{ is type II} & \iff \text{ all } U_x \text{ distinct for } x \in G/H
\end{align*}
\]
Theorem 5.

(i) $V \in \hat{G}$ is type I $\implies \text{Res}_H^G V = U \in \hat{H}$ is type I and \( \text{Ind}_H^G U \cong \bigoplus_{k \in \mathbb{Z}/p} (V \otimes \Omega(k)) \)

(ii) $V \in \hat{G}$ is type II $\implies \text{Res}_H^G V = \bigoplus_{x \in G/H} U_x$ with each $U_x$ irreducible of type II and \( \text{Ind}_H^G U_x \cong V \) for all $x \in G/H$

Proof. We will suppress stating the groups involved when they are understood, so $\text{Res}_H^G V$ and $\text{Ind}_H^G U$ become Res $V$ and Ind $U$ respectively.

Outline of Proof:
1. Show any $V \in \hat{G}$ restricts to either a type I or type II $H$-module.
2. If Res $V$ is type I, then $V$ must have been type I and $\text{Ind}_H^G U \cong \bigoplus_{k \in \mathbb{Z}/p} (V \otimes \Omega(k))$.
3. If Res $V$ is type II, then $V$ must have been type II and $\text{Ind}_H^G U_x \cong V$ for all $x \in G/H$.

Let $V \in \hat{G}$. Consider Res $V$. Either this is irreducible or not. If Res $V = U$ is an irreducible $H$-module, then it is type I since $U_x = \text{Res}_V x \cong \text{Res} V$. Now, suppose Res $V$ is reducible. Then, Res $V = U_1 \oplus \ldots \oplus U_j$ for some $j > 1$ where each $U_i$ is an irreducible $H$-module. We have the following:

\[
\bigoplus_{k \in \mathbb{Z}/p} (V \otimes \Omega(k)) \cong \text{Ind}(\text{Res} V) \quad \text{by Lemma 3}
\cong \text{Ind}(U_1 \oplus \ldots \oplus U_j)
\cong \bigoplus_{i=1}^j \text{Ind} U_i
\]

Recall $V$ and $\Omega(k)$ are in $\hat{G}$, and hence $V \otimes \Omega(k)$ is in $\hat{G}$ too. Thus, there exists a $k_0$ such that $V \otimes \Omega(k_0) \subseteq \text{Ind} U_1$ in particular. Then, it follows:

\[
\implies \text{Res}(V \otimes \Omega(k_0)) \subseteq \text{Res}(\text{Ind} U_1) = \bigoplus_{x \in G/H} (U_1)_x \quad \text{by Lemma 2}
\implies \text{Res} V \subseteq \bigoplus_{x \in G/H} (U_1)_x
\implies U_1 \oplus \ldots \oplus U_j \subseteq U_{x_1} \oplus \ldots \oplus U_{x_p} \quad \text{denoting } U = U_1 \text{ and } x_i \in \mathbb{Z}/p
\]

Our second implication above follows since restricting a representation of $G/H$ to $H$ simply yields the trivial representation. Now, since the $U_i$ and $U_{x_i}$ are all irreducible $H$-modules, we
know that each $U_i$ is contained in some $U_{x_j}$. Hence, all $U_i$ are twisted $H$-modules. Observe too that $\text{Res}\, V \cong \text{Res}\, V_x$. That is, $U_1 \oplus \ldots \oplus U_j \cong (U_1 \oplus \ldots \oplus U_j)_x \cong (U_1)_x \oplus \ldots \oplus (U_j)_x$ for all $x$. So each $U_x$ for $x \in G/H$ must appear among the $U_i$ list.

Now $\text{Ind}(\text{Res}\, V)$ splits into $p$ irreducible summands. Say by way of contradiction that $\text{Res}\, V \not\cong \bigoplus_{x \in G/H} U_x$. Then, $\text{Res}\, V = pU$ where $U$ is type I. Then,

$$p^2 = \langle pU, pU \rangle_H = \langle \text{Res}\, V, \text{Res}\, V \rangle_H = \langle \text{Ind}(\text{Res}\, V), V \rangle_G = \sum_{k=0}^{p-1} \langle V \otimes \Omega(k), V \rangle_G \leq p$$

gives a contradiction. Thus, $\text{Res}\, V$ is reducible, and $\text{Res}\, V \cong \bigoplus_{x \in G/H} U_x$. Hence, part 1 of the proof outline is shown.

**Show**(i)
Suppose $V \in \hat{G}$ gives $\text{Res}\, V = U$ irreducible. Observe

$$1 = \langle \text{Res}\, V, \text{Res}\, V \rangle_H = \langle \text{Ind}(\text{Res}\, V), V \rangle_G = \sum_{k=0}^{p-1} \langle V \otimes \Omega(k), V \rangle_G$$

and thus, all $V \otimes \Omega(k)$ must be distinct and hence $V$ is type I. Lastly, $U = \text{Res}\, V$ implies $\text{Ind}\, U = \text{Ind}(\text{Res}\, V) = \bigoplus_{k \in \mathbb{Z}/p} V \otimes \Omega(k)$ as desired and (i) is shown.

**Show**(ii)
Suppose $V \in \hat{G}$ gives $\text{Res}\, V = \bigoplus_{x \in G/H} U_x$. Then

$$p = \langle \bigoplus U_x, \bigoplus U_x \rangle_H = \langle \text{Res}\, V, \text{Res}\, V \rangle_H = \langle \text{Ind}(\text{Res}\, V), V \rangle_G = \sum_{k=0}^{p-1} \langle V \otimes \Omega(k), V \rangle_G .$$

Hence, all the $V \otimes \Omega(k)$ are isomorphic and $V$ is of type II. We now claim $\text{Ind}_H^G U_x \cong V$ for all $x \in G/H$. We have $\text{Res}\, V = \bigoplus_{x \in G/H} U_x$. This implies

$$\bigoplus_k V \cong \bigoplus_k V \otimes \Omega(k) \cong \text{Ind}(\text{Res}\, V) = \text{Ind} \left( \bigoplus_{x \in G/H} U_x \right) = \bigoplus_{x \in G/H} \text{Ind} U_x$$

And hence $\text{Ind} U_x = V$ for all $x \in G/H$ as desired and (ii) is shown.

We now apply this to our specific case of the groups $G = \mathfrak{S}_n$ and $H = \mathfrak{A}_n$. We get the following theorem. At this point, it is good to digress for a moment and discuss the well-known correspondence of irreducible $\mathfrak{S}_n$ representations and Young tableaux. This
bijection dates back as far as 1900 by Frobenius in [2]. Furthermore, we should remark too on this correspondence for irreducible \( \mathfrak{A}_n \) representations. A complete set of irreducible representations of \( \mathfrak{S}_n \) is given by \( \{ S^{(n)}_\lambda \mid \lambda \vdash n \} \). From this set, one can deduce that a complete set of irreducible representations of \( \mathfrak{A}_n \) is given by \( \{ A^{(n)}_\lambda \mid \lambda \neq \lambda' \} \cup \{ A^{(n)}_{\lambda^+}, A^{(n)}_{\lambda^-} \mid \lambda = \lambda' \} \) where \( A^{(n)}_{\lambda^+} \) and \( A^{(n)}_{\lambda^-} \) are conjugate representations of \( \mathfrak{A}_n \). That is, \( (A^{(n)}_{\lambda^+})(12) \) defined by \( (A^{(n)}_{\lambda^+})(12)((12)\pi(12)) := A^{(n)}_{\lambda^+}(\pi) \) for \( \pi \in A_n \) is equivalent to \( A^{(n)}_{\lambda^+}(\pi) \). See [3, pg 66] for details.

**Theorem 6** (Clifford Theory for \( \mathfrak{S}_n \)). Let \( S^{(n)}_\lambda \) be an irreducible \( \mathfrak{S}_n \)-module and \( A^{(n)}_\lambda \) be an irreducible \( \mathfrak{A}_n \)-module. Then the following equalities hold:

\[
\begin{align*}
\text{Res}_{\mathfrak{A}_n} S^{(n)}_\lambda &\cong \begin{cases} 
A^{(n)}_\lambda & \text{if } \lambda \nmid \lambda' \\
A^{(n)}_{\lambda^+} & \text{if } \lambda \cong \lambda'
\end{cases} \\
\text{Ind}_{\mathfrak{A}_n} A^{(n)}_\lambda &\cong \begin{cases} 
S^{(n)}_\lambda & \text{if } \lambda \nmid \lambda' \\
S^{(n)}_{\lambda^+} & \text{if } \lambda \cong \lambda'
\end{cases}
\end{align*}
\]

**Proof.** This is a direct application of Theorem 5 above. In the latter theorem, simply let \( G = \mathfrak{S}_n \), \( H = \mathfrak{A}_n \), and \( V = S^{(n)}_\lambda \). Then since \( [G : H] = 2 \), we have \( k = 2 \) and the case of \( V \in \hat{G} \) being type I corresponds to \( V \cong V' \) (where \( V' \) is the conjugate representation to \( V \)). That is, \( V := V \otimes \Omega(0) \) and \( V' := V \otimes \Omega(1) \) from Theorem 5 are distinct. Similarly, the case of \( V \in \hat{G} \) being type II corresponds to \( V \cong V' \). That is, \( V \otimes \Omega(0) \) and \( V \otimes \Omega(1) \) are isomorphic. So making the appropriate symbol replacements, we may rewrite the statement of Theorem 5 as the following:

\[
V \ncong V' \implies \text{Res} \, V = \text{Res} \, V' = U \text{ and } \text{Ind} \, U \cong V \oplus V' \tag{3}
\]

\[
V \cong V' \implies \text{Res} \, V = W^+ \oplus W^- \text{ and } \text{Ind} \, W^+ \cong \text{Ind} \, W^- \cong V \tag{4}
\]

The symbols \( W^+ \) and \( W^- \) are simply \( U_0 \) and \( U_1 \) from our application of Theorem 5 noting in this index two case that \( x \in G/H \cong \mathbb{Z}/2 \). Using (3), we notice that \( U := A^{(n)}_\lambda \) is a type I irreducible \( \mathfrak{A}_n \)-module, and we get the first case in (1) above. By the second conclusion of (3), we get the first case of (2). Similarly the first conclusion of (4) yields the second case of (1) above, and the second conclusion of (4) yields the second case of (2) above as desired. This proves the theorem. \( \square \)

### 2.3 Deducing the branching rules for \( \mathfrak{A}_n \)

We will utilize the known branching rules for the symmetric group in Theorem 1 and our recent result on Clifford theory for \( \mathfrak{S}_n \) in Theorem 6 above to deduce the branching rules for
\( A_n \). The proof below involves only these two known results and a little Frobenius reciprocity and Mackey theory as you will see.

**Theorem 7 (\( A_n \) Branching Rules).** Let \( A^{(n)}_\lambda \) be an irreducible \( A_n \)-module and \( A^{(n-1)}_\mu \) be an irreducible \( A_{n-1} \)-module. Then

\[
\text{Res}_{A_{n-1}} A^{(n)}_\lambda \cong \begin{cases} \sum_{\mu = \lambda - \boxplus \mu' \neq \mu} A^{(n-1)}_\mu \oplus A^{(n-1)}_{\mu'} & \text{if } \lambda \neq \lambda' \\ \sum_{\mu = \lambda - \boxplus \mu' \neq \mu} A^{(n-1)}_\mu & \text{if } \lambda = \lambda' \end{cases}
\]

(5)

\[
\text{Ind}_{A_{n-1}} A^{(n-1)}_\mu \cong \begin{cases} \sum_{\lambda = \mu + \boxplus \lambda' \neq \lambda} A^{(n)}_\lambda \oplus A^{(n)}_{\lambda'} & \text{if } \mu \neq \mu' \\ \sum_{\lambda = \mu + \boxplus \lambda' \neq \lambda} A^{(n)}_\lambda \oplus A^{(n)}_{\lambda'} & \text{if } \mu = \mu' \end{cases}
\]

(6)

where condition \( \star \) means do not include conjugate repeats, for example, the tableau \((2,1,1)\) is the same as \((3,1)\) in this alternating group case.
Proof.

Show \( \lambda \neq \lambda' \) case of (5):

If \( \lambda \neq \lambda' \), then the following string of equalities hold:

\[
\text{Res}_{A_n}^\mathfrak{A}_{n-1} A_{n}^{(n)} \cong \text{Res}_{\mathfrak{A}_{n-1}}^\mathfrak{S}_{n-1} \left( \text{Res}_{\mathfrak{S}_{n}}^\mathfrak{A}_{n} S_{\lambda}^{(n)} \right) \\
= \text{Res}_{\mathfrak{A}_{n-1}}^\mathfrak{S}_{n-1} \left( \text{Res}_{\mathfrak{S}_{n-1}}^\mathfrak{S}_{n} S_{\lambda}^{(n)} \right) \\
\cong \text{Res}_{\mathfrak{A}_{n-1}}^\mathfrak{S}_{n-1} \left( \bigoplus_{\mu = \lambda - \square} S_{\mu}^{(n-1)} \right) \\
= \bigoplus_{\mu = \lambda - \square} \left( \text{Res}_{\mathfrak{A}_{n-1}}^\mathfrak{S}_{n-1} S_{\mu}^{(n-1)} \right) \\
\cong \text{Res}_{\mathfrak{A}_{n-1}}^\mathfrak{S}_{n-1} \left( \bigoplus_{\mu = \lambda - \square} A_{\mu}^{(n-1)} \right) \oplus \sum_{\mu = \lambda - \square} \left( A_{\mu_+}^{(n-1)} \oplus A_{\mu_-}^{(n-1)} \right) \\
\text{by Clifford theory (Theorem 6)}
\]

Now we note an observation about the second sum in the last equality. Since \( \lambda \neq \lambda' \), we say \( \lambda \) is not self-conjugate. It is easily observed that the set \( \{ \mu = \lambda - \square \} \) can yield at most one self-conjugate tableau. Hence, our second sum in the last equality has only one summand and we conclude

\[
\text{Res}_{\mathfrak{A}_{n-1}}^\mathfrak{S}_{n} A_{n}^{(n)} \cong \left( \bigoplus_{\mu = \lambda - \square} A_{\mu}^{(n-1)} \right) \oplus \left( A_{\mu_+}^{(n-1)} \oplus A_{\mu_-}^{(n-1)} \right)
\]

as desired.
Show $\lambda = \lambda'$ case of (5):
If $\lambda = \lambda'$, then we have a self-conjugate tableau, and hence our irreducible $\mathfrak{A}_n$-module $A^{(n)}_\lambda$ is either of the type $A^{(n)}_{\lambda^+}$ or $A^{(n)}_{\lambda^-}$. Let us consider $\text{Res}_{\mathfrak{A}_n}^{\mathfrak{A}_{n-1}} A^{(n)}_{\lambda^+} \oplus \text{Res}_{\mathfrak{A}_n}^{\mathfrak{A}_{n-1}} A^{(n)}_{\lambda^-}$ noting that this equals $\text{Res}_{\mathfrak{A}_n}^{\mathfrak{A}_{n-1}} \left(A^{(n)}_{\lambda^+} \oplus A^{(n)}_{\lambda^-}\right)$. Then we have:

$$
\text{Res}_{\mathfrak{A}_n}^{\mathfrak{A}_{n-1}} \left(A^{(n)}_{\lambda^+} \oplus A^{(n)}_{\lambda^-}\right) \cong \text{Res}_{\mathfrak{A}_n}^{\mathfrak{A}_{n-1}} \left(\text{Res}_{\mathfrak{S}_n}^{\mathfrak{S}_{n-1}} S^{(n)}_{\lambda}\right)
$$

by Clifford theory (Theorem 6)

$$
= \text{Res}_{\mathfrak{A}_n}^{\mathfrak{A}_{n-1}} \left(\text{Res}_{\mathfrak{S}_n}^{\mathfrak{S}_{n-1}} S^{(n)}_{\lambda}\right)
$$

by path change from $\mathfrak{S}_n$ to $\mathfrak{A}_{n-1}$

$$
\cong \text{Res}_{\mathfrak{A}_n}^{\mathfrak{A}_{n-1}} \left(\bigoplus_{\mu = \lambda - \square} S^{(n-1)}_{\mu}\right)
$$

by $\mathfrak{S}_n$-branching (Theorem 1)

$$
= \bigoplus_{\mu = \lambda - \square} \left(\text{Res}_{\mathfrak{A}_n}^{\mathfrak{A}_{n-1}} S^{(n-1)}_{\mu}\right)
$$

Now we make a very important observation. First, let us consider the first sum in this last equality. There are two copies of each irreducible here since each occurring $\mu$ is not self-conjugate. For example, if say $\lambda$ were a $(3,1,1)$-tableau, then the two ways of removing one box would yield a $(3,1)$ and a $(2,1,1)$ tableaux. But, $A^{(n-1)}_{(3,1)}$ is EXACTLY the $A^{(n-1)}_{(2,1,1)}$ representation! Moreover, these two copies can NOT lie in both $\text{Res}_{\mathfrak{A}_n}^{\mathfrak{A}_{n-1}} A^{(n)}_{\lambda^+}$ and $\text{Res}_{\mathfrak{A}_n}^{\mathfrak{A}_{n-1}} A^{(n)}_{\lambda^-}$. For instance, if $A^{(n-1)}_{(3,1)} \in \text{Res}_{\mathfrak{A}_n}^{\mathfrak{A}_{n-1}} A^{(n)}_{\lambda^+}$, then $A^{(n-1)}_{(2,1,1)} \in \text{Res}_{\mathfrak{A}_n}^{\mathfrak{A}_{n-1}} A^{(n)}_{\lambda^-}$ necessarily. Lastly, our same observation from the case above holds for the second sum in the last equality above. So this second sum becomes simply a single summand $A^{(n-1)}_{\mu^+} \oplus A^{(n-1)}_{\mu^-}$. Observe, $A^{(n-1)}_{\mu^+}$ lies in $\text{Res}_{\mathfrak{A}_n}^{\mathfrak{A}_{n-1}} A^{(n)}_{\lambda^+}$ or $\text{Res}_{\mathfrak{A}_n}^{\mathfrak{A}_{n-1}} A^{(n)}_{\lambda^-}$ but not both since it is irreducible. Without loss of generality, we will say $A^{(n-1)}_{\mu^+}$ sits in $\text{Res}_{\mathfrak{A}_n}^{\mathfrak{A}_{n-1}} A^{(n)}_{\lambda^+}$. Hence we conclude

$$
\text{Res}_{\mathfrak{A}_n}^{\mathfrak{A}_{n-1}} A^{(n)}_{\lambda} \cong \left(\sum_{\mu = \lambda - \square, \mu \neq \mu'} A^{(n-1)}_{\mu}\right) \bigoplus_{\mu = \lambda - \square} A^{(n-1)}_{\mu}\hspace{0.5cm}\text{condition } \star
$$

where in this right-most summand, we choose the positive (resp., negative) $\mu$ if $\lambda$ were positive (resp., negative).

Show $\mu \neq \mu'$ case of (6):
First we prove a useful claim using Mackey theory. We would like to show the following

\[
\text{Ind}_{\mathfrak{A}_{n-1}}^{\mathfrak{A}_n} \left( \text{Res}_{\mathfrak{A}_{n-1}}^{\mathfrak{A}_n} S^{(n-1)}_{\mu} \right) \cong \text{Res}_{\mathfrak{A}_n}^{\mathfrak{S}_n} \left( \text{Ind}_{\mathfrak{A}_{n-1}}^{\mathfrak{S}_n} S_{\mu}^{(n-1)} \right)
\]

Recall in general, for subgroups \( H, K \leq G \) and \( V \) a representation of \( K \) over a commutative ring \( R \), Mackey theory gives that \( \text{Res}_H^G(\text{Ind}_K^G V) \cong \bigoplus_{g \in [H \backslash G / K]} \text{Ind}_{H \cap g K}^H (g(R_{H \cap g K} V)) \) are isomorphic as \( R[H] \)-modules. Following Webb’s notation in his free online text [4, pg 58], we have \( gK \) is the left conjugate representation of \( K \) and \( Hg \) the right conjugate representation of \( H \) and \( g \in [H \backslash G / K] \) means let \( g \) run over a set of \((H, K)\)-double cosets representatives for \( H \backslash G / K \). Applying this to our particular case, we have \( H = \mathfrak{A}_n, G = \mathfrak{S}_n, K = \mathfrak{S}_{n-1}, \) and \( V = S_{\mu}^{(n-1)} \). So Mackey’s theorem yields that

\[
\text{Res}_{\mathfrak{A}_n}^{\mathfrak{S}_n} \left( \text{Ind}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} S^{(n-1)}_{\mu} \right) \cong \bigoplus_{g \in [\mathfrak{A}_n \backslash \mathfrak{S}_n / \mathfrak{S}_{n-1}]} \text{Ind}_{\mathfrak{A}_n \cap g \mathfrak{S}_{n-1}}^{\mathfrak{A}_n \cap g \mathfrak{S}_n} \left( g(R_{\mathfrak{A}_n \cap g \mathfrak{S}_{n-1}} S^{(n-1)}_{\mu}) \right)
\]

(7)

We claim that \( \mathfrak{A}_n \backslash \mathfrak{S}_n / \mathfrak{S}_{n-1} \) has only one double coset. In [4, pg 56], it is shown that the set of \((H, K)\)-double cosets are in bijection to the orbits of \((H \backslash G) / K \). In our case, since \( \mathfrak{A}_n \backslash \mathfrak{S}_n = \{ \mathfrak{A}_n e, \mathfrak{A}_n (12) \} \), clearly there is the transposition \((12) \in \mathfrak{S}_{n-1} \) (noting \( n > 2 \)) which can take us back and forth between the two cosets of \( \mathfrak{A}_n \backslash \mathfrak{S}_n \). Thus, \( \mathfrak{A}_n \backslash \mathfrak{S}_n / \mathfrak{S}_{n-1} \) has only one double coset, and our representative \( g \) from (7) above must be the identity \( e \in \mathfrak{S}_n \). Then, it clearly follows that \( \mathfrak{A}_n \cap g \mathfrak{S}_{n-1} = \mathfrak{A}_{n-1} \) and \( \mathfrak{A}_n \cap \mathfrak{S}_{n-1} = \mathfrak{A}_{n-1} \) since \( g = e \). Making the replacements in (7) above, we get

\[
\text{Res}_{\mathfrak{A}_n}^{\mathfrak{S}_n} \left( \text{Ind}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} S^{(n-1)}_{\mu} \right) \cong \text{Ind}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} \left( \text{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} S^{(n-1)}_{\mu} \right)
\]

which proves our claim on Mackey theory.

Now to show the first case of (6), we have the following string of equalities:

\[
\text{Ind}_{\mathfrak{A}_{n-1}}^{\mathfrak{A}_n} A^{(n-1)}_{\mu} \cong \text{Ind}_{\mathfrak{A}_{n-1}}^{\mathfrak{A}_n} \left( \text{Res}_{\mathfrak{A}_{n-1}}^{\mathfrak{A}_n} S^{(n-1)}_{\mu} \right) \quad \text{by Clifford theory (Theorem 6)}
\]

\[
\cong \text{Res}_{\mathfrak{A}_n}^{\mathfrak{S}_n} \left( \text{Ind}_{\mathfrak{A}_{n-1}}^{\mathfrak{S}_n} S^{(n-1)}_{\mu} \right) \quad \text{by Mackey result above}
\]

\[
\cong \text{Res}_{\mathfrak{A}_n}^{\mathfrak{S}_n} \left( \bigoplus_{\lambda = \mu + \Box} S^{(n)}_{\lambda} \right) \quad \text{by } \mathfrak{S}_n \text{ branching rules (Theorem 1)}
\]

\[
= \bigoplus_{\lambda = \mu + \Box} \left( \text{Res}_{\mathfrak{A}_n}^{\mathfrak{S}_n} S^{(n)}_{\lambda} \right)
\]

\[
\cong \left( \sum_{\lambda = \mu + \Box} A^{(n)}_{\lambda} \right) \oplus \sum_{\lambda = \mu + \Box} \left( A^{(n)}_{\lambda^+} \oplus A^{(n)}_{\lambda^-} \right) \quad \text{by Clifford theory (Theorem 6)}
\]
In a similar manner to our first case, we make the observation that adding a box in every way possible to a non-self-conjugate tableau can yield at most one self-conjugate tableau. Hence our second sum in the last equality has only one summand, so we conclude

\[
\Ind_{\mathfrak{A}_{n-1}}^{\mathfrak{A}_n} A_{(n-1)}^{(n)} \cong \left( \sum_{\lambda=\mu+ \square, \lambda \neq \lambda'} A_{(n)}^{(n)} \right) \bigoplus_{\lambda=\mu+ \square} \left( A_{(n)}^{(n)} \oplus A_{(n)}^{(n)} \right)
\]

Show \( \mu = \mu' \) case of (6):
We claim that Frobenius reciprocity gives us this for free using:

\[
\left( A_{(n)}^{(n)}, \Ind_{\mathfrak{A}_{n-1}}^{\mathfrak{A}_n} A_{(n-1)}^{(n)} \right)_{\mathfrak{A}_n} = \left( \Res_{\mathfrak{A}_{n-1}}^{\mathfrak{A}_n} A_{(n)}^{(n)}, A_{(n-1)}^{(n)} \right)_{\mathfrak{A}_{n-1}}
\]

Consider the \( \lambda \) constructed by adding a box to \( \mu \). Either this \( \lambda \) is self-conjugate or not. If \( \lambda \neq \lambda' \), then we have:

\[
\left( A_{(n)}^{(n)}, \Ind_{\mathfrak{A}_{n-1}}^{\mathfrak{A}_n} A_{(n-1)}^{(n)} \right)_{\mathfrak{A}_n} = \left( \left( \sum_{\nu = \lambda - \square, \nu \neq \nu'} A_{(n-1)}^{(n-1)} \right) \bigoplus_{\nu = \lambda - \square, \nu \neq \nu'} \left( A_{(n-1)}^{(n-1)} \oplus A_{(n-1)}^{(n-1)} \right), A_{(n)}^{(n)} \right)
\]

Taking a look at the summands above, each summand of \( \sum_{\nu \neq \nu'} A_{(n-1)}^{(n-1)} A_{(n)}^{(n-1)} \) is clearly zero since \( \nu \) is not self-conjugate but \( \mu^+ \) is. The last summand is clearly zero since \( \nu^- \) and \( \mu^+ \) have differing polarities. Lastly, the middle summand \( A_{(n-1)}^{(n-1)} A_{(n-1)}^{(n-1)} \) is 1 if \( \nu^+ = \mu^+ \) and 0 otherwise.

Now suppose \( \lambda = \lambda' \). Then we have:

\[
\left( A_{(n)}^{(n)}, \Ind_{\mathfrak{A}_{n-1}}^{\mathfrak{A}_n} A_{(n-1)}^{(n-1)} \right)_{\mathfrak{A}_n} = \left( \left( \sum_{\nu = \lambda - \square, \nu \neq \nu'} A_{(n-1)}^{(n-1)} \right) \bigoplus_{\nu = \lambda - \square, \nu \neq \nu'} \left( A_{(n-1)}^{(n-1)} \oplus A_{(n-1)}^{(n-1)} \right), A_{(n)}^{(n)} \right)
\]

Again all the summands in \( \sum_{\nu \neq \nu'} A_{(n-1)}^{(n-1)} A_{(n-1)}^{(n-1)} \) are zero since \( \nu \) is not self-conjugate but \( \mu^+ \) is. And the last summand will yield 1 only when \( \nu^\pm = \mu^+ \). This proves the \( \mu = \mu' \) case of (6), and we are done.

\[\Box\]
References


