The gamma function ($\Gamma$) is an extension of the factorial function to $\mathbb{C} \setminus \mathbb{Z}^- \cup 0$.

It is a solution to the interpolation problem of connecting the discrete points of the factorial function.
What is the Gamma Function?

Definitions

Factorial function definition

If $n \in \mathbb{Z}^+$, then $\Gamma(n) = (n-1)!$

Improper integral definition

If $z \in \{x + iy \mid x > 0\}$, then $\Gamma(z) = \int_{0}^{\infty} x^{z-1} e^{-x} \, dx$.

- Analytic continuation is required to extend the integral definition to $z \in \{x + iy \mid y \neq 0 \text{ when } x \in \mathbb{Z}^- \cup 0\}$.
- This function has simple poles at all the non-positive integers.
More Definitions

Definition as an infinite product

\[ \Gamma(z) = \frac{1}{z} \prod_{n=1}^{\infty} \frac{(1 + \frac{1}{n})^z}{1 + \frac{z}{n}} \]

Weierstrass’s definition

\[ \Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}ight)^{-1} e^{z/n} \]
Identities and Formulas

- Requirements for $\Gamma$ to be an extension of the factorial function:
  \[ \Gamma(1) = 1 \]
  \[ z\Gamma(z) = \Gamma(z + 1) \text{ for } \Re(z) > 0 \]

- Property of conjugation:
  \[ \overline{\Gamma(z)} = \Gamma(\overline{z}) \]

- Complement Formula:
  \[ \Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin(\pi z)} \text{ for } z \notin \mathbb{Z} \]

- Duplication Formula:
  \[ \Gamma(z)\Gamma(z + \frac{1}{2}) = \Gamma(2z) \cdot 2^{1-2z} \sqrt{\pi} \]
Proof of $z\Gamma(z) = \Gamma(z + 1)$

Let $\alpha \in \mathbb{C}$ such that $Re(\alpha) > 0$

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} \, dx$$
$$= x^{\alpha-1}(-e^{-x})\bigg|_0^\infty - \int_0^\infty -e^{-x}(\alpha - 1)x^{\alpha-2} \, dx$$
$$= -\lim_{x \to \infty} \frac{x^{\alpha-1}}{e^x} + \int_0^\infty e^{-x}(\alpha - 1)x^{\alpha-2} \, dx$$
$$= 0 + (\alpha - 1) \int_0^\infty e^{-x}x^{\alpha-1-1} \, dx$$

$$\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$$

Take $z = \alpha - 1$.

$$\Gamma(z + 1) = z\Gamma(z)$$
Values of the Gamma Function

\[ \Gamma(1) = 1 \]
\[ \Gamma(2) = 1 \]
\[ \Gamma(3) = 2 \]
\[ \Gamma(4) = 6 \]
\[ \Gamma(5) = 24 \]

\[ \Gamma\left(-\frac{3}{2}\right) = \frac{4}{3} \sqrt{\pi} \]
\[ \Gamma\left(-\frac{1}{2}\right) = -2 \sqrt{\pi} \]
\[ \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \]
\[ \Gamma\left(\frac{3}{2}\right) = \frac{1}{2} \sqrt{\pi} \]
\[ \Gamma\left(\frac{5}{2}\right) = \frac{3}{4} \sqrt{\pi} \]
\[ \Gamma\left(\frac{7}{2}\right) = \frac{15}{8} \sqrt{\pi} \]
Example: Proving \((\frac{1}{2})! = \frac{\sqrt{\pi}}{2}\)

\[
\left(\frac{1}{2}\right)! = \Gamma \left(\frac{3}{2}\right)
\]

\[
= \frac{1}{2} \Gamma \left(\frac{1}{2}\right)
\]

\[
= \frac{1}{2} \int_{0}^{\infty} t^{-1/2} e^{-t} \, dt
\]

Let \(u = t^{1/2}\)

Then \(du = \frac{1}{2} t^{-1/2} \, dt\)

\[
= \int_{0}^{\infty} e^{-u^2} \, du
\]

\[
= \frac{\sqrt{\pi}}{2}
\]
The log-gamma function

The Gamma function grows rapidly, so taking the natural logarithm yields a function which grows much more slowly:

$$\ln \Gamma(z) = \ln \Gamma(z + 1) - \ln z$$

This function is used in many computing environments and in the context of wave propagation.
The Digamma function is defined to be the logarithmic derivative of the Gamma function:

\[ \psi(z) = \frac{d}{dz} \ln(\Gamma(z)) = \frac{\Gamma'(z)}{\Gamma(z)} \]

A general form of the Digamma function: the Polygamma function
- defined to be the \((m + 1)th\) logarithmic derivative of the Gamma function:
  \[ \psi^{(m)}(z) = \frac{d^m}{dz^m} \ln(\Gamma(z)) \]
- Notice that \(\psi^{(0)}(z) = \psi(z)\)
- The Polygamma function is meromorphic on \(\mathbb{C}\) (holomorphic on \(\mathbb{C} \setminus \mathbb{Z}^- \cup 0\))
- The nonpositive integers have poles of order \(m + 1\)
Digamma and Polygamma functions

\[ \ln \Gamma(z) \]
\[ \psi^{(0)}(z) \]
\[ \psi^{(1)}(z) \]
\[ \psi^{(2)}(z) \]
\[ \psi^{(3)}(z) \]
\[ \psi^{(4)}(z) \]
Digamma and Polygamma functions

Finding a property of the Digamma function:
Recall that $\Gamma(z + 1) = z\Gamma(z)$

Take the derivative:

$$\Gamma'(z + 1) = z\Gamma'(z) + \Gamma(z)$$

Divide by $\Gamma(z + 1) = z\Gamma(z)$:

$$\frac{\Gamma'(z + 1)}{\Gamma(z + 1)} = \frac{\Gamma'(z)}{\Gamma(z)} + \frac{1}{z}$$

Substitute in $\psi$ function:

$$\psi(z + 1) = \psi(z) + \frac{1}{z}$$
Incomplete Gamma Functions

upper incomplete gamma function

\[ \Gamma(s, x) = \int_x^\infty x^{s-1} e^{-t} \, dt \]

lower incomplete gamma function

\[ \gamma(s, x) = \int_0^x x^{s-1} e^{-t} \, dt \]
Pi and Beta Functions

Definition of the Beta Function

\[ B(x, y) = \int_0^1 t^{x-1}(1 - t)^{y-1} \, dt \text{ for } \Re(x), \Re(y) > 0 \]

Beta function in terms of the Gamma function:

\[ B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)} \]

Definition of the Pi Function

\[ \Pi(z) = \Gamma(z + 1) = z\Gamma(z) \]
Riemann Zeta Function

The functional equation:

\[ \zeta(s) = 2^s \pi^{s-1} \sin \frac{\pi s}{2} \Gamma(1 - s) \zeta(1 - s) \]

The functional equation in another form:

\[ \pi^{-\frac{s}{2}} \Gamma \left( \frac{s}{2} \right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma \left( \frac{1-s}{2} \right) \zeta(1 - s) \]

Another relation:

\[ \zeta(z) \Gamma(z) = \int_0^\infty \frac{u^{z-1}}{e^u - 1} \, du \text{ for } \text{Re}(z) > 1 \]