# Part III: The Riemann Spectrum of the Prime Numbers

Geoffrey Glover    Grant Keane    Rita Post

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# Recall...Staircase of Primes

Staircase of primes $\pi(X)$: the red curve below represents the accumlation of primes up to $X$.

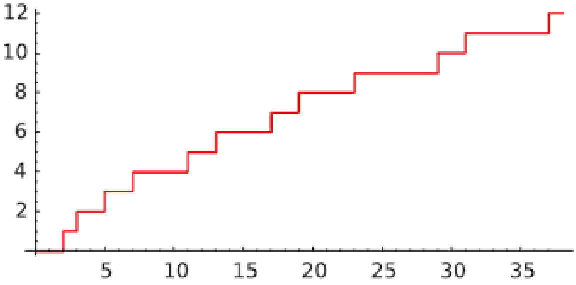



Figure 31.2. Prime numbers up to 37

After looking at the staircase of primes, we want to construct $\pi(X)$ from the **spectrum of primes** $\theta_1, \theta_2, \ldots$ (i.e., the non-trivial zeros of the Riemann-zeta function).

# Where we're going

Recall Chebychev's weighted prime counting function

$$\psi(x) = \sum_{p^m \leq x} \log p.$$

We will be replacing this function with a generalized function

$$\Phi(t) = e^{-\frac{t}{2}} \psi'(t)$$

that has support at all positive integral multiples of logs of prime numbers.

Why construct $\Phi(t)$?

1. To contain all valuable information of $\psi(X)$, including the placement of primes among numbers.
2. To pay close attention to the spike values of the trigonometric series that is the Fourier transform of $\Phi(t)$.

# Fourier Transforms

Recall that the **Dirac delta function** $\delta_0(t)$ equals infinity when $t = 0$ and zero when $t \neq 0$,

Moreover the **Fourier transform of the delta function** is

$$\widehat{\delta_0}(\theta) = \int_{-\infty}^{\infty} \cos(\theta t)\, \delta_0(t) dt.$$

Also recall the **spike function** is

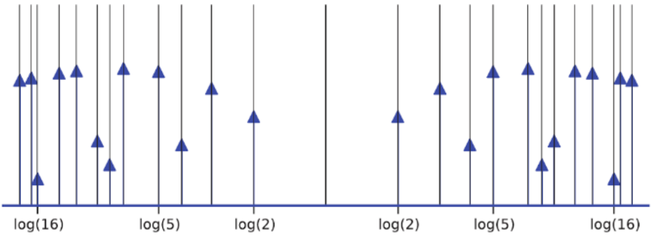
$$d_x(t) = \frac{(\delta_x(t) + \delta_{-x}(t))}{2},$$

and the **Fourier transform of spike function** is

$$\widehat{d}_x(\theta) = \cos(x\theta).$$

# A graph of $\Phi(t)$

$\Phi(t)$ is a sum of Dirac delta functions at the logarithms of prime powers $p^n$ weighted by $p^{\frac{-n}{2}} \log(p)$.



# Another way to think of $\Phi(t)$

$\Phi(t)$ can be observed as a limit of a sequence of distributions,

$$\Phi(t) = \lim_{C \to \infty} \Phi_{\leq C}(t)$$

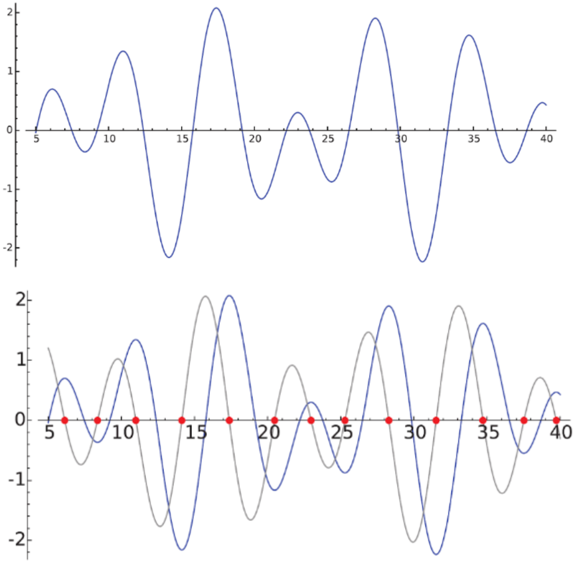
where $\Phi_{\leq C}(t)$ is the following finite linear combination of $\delta$-functions of $d_x(t)$

$$\Phi_{\leq C}(t) = 2 \sum_{p^n \leq C} p^{-\frac{n}{2}} \log(p) d_{n \log(p)}(t)$$

Using the spike function, it follows that the Fourier transform of the function is

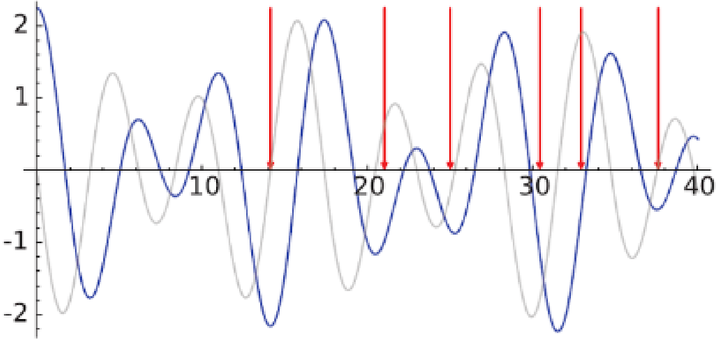
$$\widehat{\Phi}_{\leq C}(\theta) = 2 \sum_{p^n \leq C} p^{-\frac{n}{2}} \log(p) \cos(n \log(p))(\theta)$$

# Plot of $\widehat{\Phi}_{\leq 3}(\theta)$ and its derivative



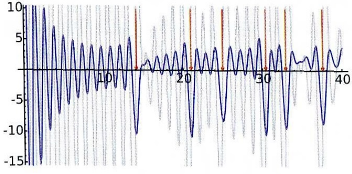
# "limit" $\theta$ values

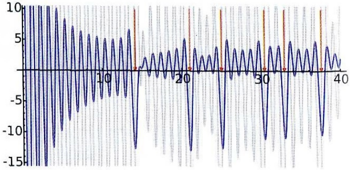
As $C$ tends to infinity, there seems to be an eventual convergence of the values of $\theta$ that correspond to higher and higher peaks.



# The Riemann spectrum of primes

If the Riemann Hypothesis holds, these numbers would be the key to the placement of primes on the number line.



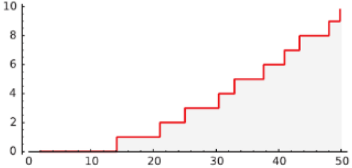

By tabulating these peaks we can approximately compute

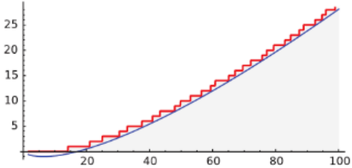
$$\theta_1 = 14.134725,\ \theta_2 = 21.022039,\ \theta_3 = 25.010857$$
$$\theta_4 = 30.424876,\ \theta_5 = 32.935061,\ \theta_6 = 37.586178$$

These $\theta_i$ appear as “imaginary parts of the nontrivial zeroes of Riemann’s zeta function.”

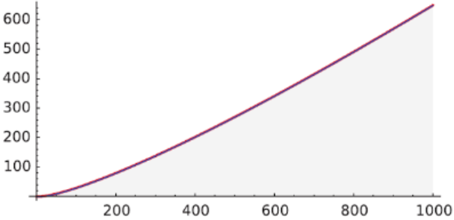
# The Staircase of the Riemann spectrum

A count of the $\theta_i$'s form the staircase on the left and is closely approximated to the curve $\frac{T}{2\pi}\log(\frac{T}{2\pi e})$, as shown to the right.



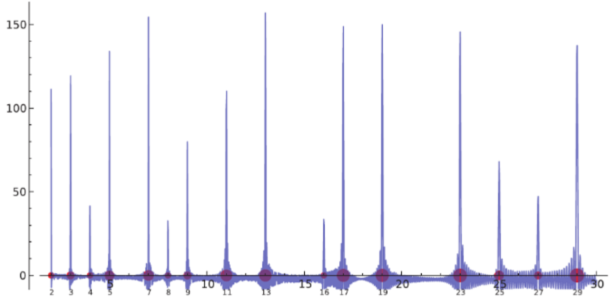




The goal is to obtain a smooth curve over a distance,as shown.



# The Riemann Spectrum of Primes

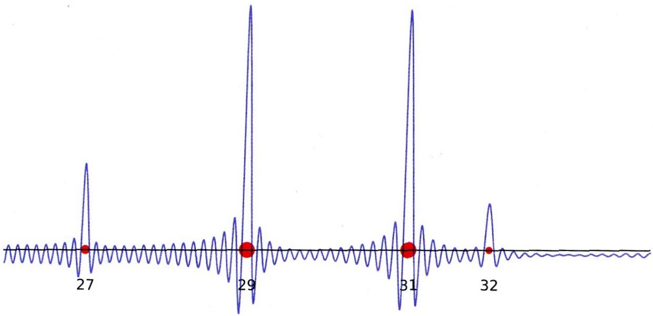
**Question:** Can we use this spectrum to get information about the placement of prime numbers?




Above is the graph of $-\sum_{i=1}^{1000} \cos(\log(s)\theta_i)$ where $\theta_i$ are the first 1000 values in the Riemann spectrum. The **red dots** are at the prime powers $p^n$ whose size is proportional to $\log(p)$.

# Connection to Twin Primes

Some twin primes are separated out by the Fourier series.




Above is the graph of $-\sum_{i=1}^{1000} \cos(\log(s)\theta_i)$ in the neighborhood of a twin prime.

# Intro to Part IV: Riemann's Smooth Curve $R(X)$

Riemann's smooth curve $R(X)$ is the fundemental approximation to $\pi(X)$ and is shown below.

$$R(X) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \operatorname{li}(X^{\frac{1}{n}}) = \lim_{n \to \infty} R^{(N)}(X) = \lim_{N \to \infty} \sum_{n=1}^{N} \frac{\mu(n)}{n} \operatorname{li}(X^{\frac{1}{n}})$$

where $\operatorname{li}(X) = \int_0^X \frac{dt}{\log(t)}$ and $\mu(n)$ where $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ is the **Möbius function** which is defined by

$$\mu(n) = \begin{cases} 1, & \text{if } \alpha_i = 1 \text{ for all } i \text{ and } k \in 2\mathbb{Z} \\ -1, & \text{if } \alpha_i = 1 \text{ for all } i \text{ and } k \in 2\mathbb{Z}+1 \\ 0, & \text{otherwise.} \end{cases}$$

# The function $R_1$ and $R_{10}$ approximating the staircase of primes up to 100