Riemann Hypothesis and Connection to the Distribution of Prime Numbers

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University of Wisconsin
Eau Claire

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Outline

1. Introduction

2. What is the Riemann hypothesis?

3. What is the Riemann-zeta function?

4. The apparent “convergence” of some divergent series?

5. Analytic continuation of the zeta function

6. Visualizing the zeros of the zeta function
How to earn $1,000,000 by taking this Math 395 course?

If you excel in this course, then you will surely get an A.
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![Image of a $100,000 bill]

**TRIVIA:** In case you’re curious, this is the largest denomination ever printed in the US. It was made only from 1934 and 1935.
The Seven Millenial Math Problems

There is a $1,000,000 prize awarded by the Clay Mathematics Institute for the solution to each of the following problems:

1. Birch and Swinnerton-Dyer conjecture
2. Hodge conjecture
3. Navier-Stokes existence and smoothness
4. P versus NP problem
5. Poincaré conjecture
6. Riemann hypothesis
7. Yang-Mills existence and mass gap

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Where is he now? He eludes the media. And he has become a recluse. No one knows if he is working on mathematics any longer or has completely abandoned it.
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Statement of the Riemann Hypothesis

Hypothesis (Riemann, 1859)

All the nontrivial zeros of $\zeta(s)$ lie on the vertical line in the complex plane consisting of the complex numbers $s = \sigma + it$ with $\sigma = \frac{1}{2}$.

(Note: The notation $s$, $\sigma$, and $t$ is that used by Riemann.)
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4. Besides analytic number theorists, WHO CARES about the Riemann hypothesis?
An answer to the question “Who cares?”

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Problem #8: The Riemann Hypothesis!
SHORT ANSWER:

Understanding the distribution of the prime numbers is directly related to understanding the zeros of the Riemann-zeta function.

Define the prime counting function by

\[ \pi(x) = \sum_{p \leq x} 1; \]

that is, the number of primes less than or equal to \( x \).

Usually we consider its weighted modification

\[ \nu(x) = \sum_{p^m \leq x} \log p \]

where we are also counting the prime powers.

It is not hard to show that

\[ \pi(x) = \nu(x) \log x + O(1) \log \log x \]

which means that these two functions differ by about a factor of \( \log x \).
“Who cares?” (continued)

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LONG ANSWER: Define the prime counting function by \( \pi(x) = \sum_{p \leq x} 1 \); that is, the number of primes less than or equal to \( x \). Usually we consider its weighted modification

\[
\psi(x) = \sum_{p^m \leq x} \log p
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\[
\pi(x) = \frac{\psi(x)}{\log x} \left( 1 + O \left( \frac{1}{\log x} \right) \right) \quad [\text{Student HW}]
\]

which means that these two functions differ by about a factor of \( \log x \).
LONG ANSWER (continued):

The prime number theorem states that \( \pi(x) \sim x / \log(x) \), but this is quite hard to show.

It was first conjectured by Legendre in 1797, but took almost 100 years to prove, resolved in 1896 by Hadamard and de la Vallée Poussin. In 1859 Riemann outlined a proof, and gave a remarkable identity which changed how people thought about counting primes. He showed

\[
\pi(x) = \frac{x}{\log(x)} - \sum_{\zeta \text{ non-trivial zeros}} \left( 1 - \frac{x}{\zeta} \right)
\]

where the sum is taken over all the nontrivial zeros of the \( \zeta \) function.

Why is that explicit formula (in blue above) miraculous? Notice that Riemann's formula for \( \pi(x) \) is an equality. The left hand side is a step function, and on the right hand side, somehow, the zeros of the \( \zeta \) function conspire at exactly the prime numbers to make that sum jump.
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$$\psi(x) = x - \sum_{\rho : \zeta(\rho) = 0} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2} \log(1 - x^{-2})$$

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Recall some Calculus II results on infinite series

1. What is the series \( \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \) called?
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4. Does the series \( \sum_{n=1}^{\infty} \frac{1}{n^3} = 1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \cdots \) converge? And if so, to what value?
Definition of the zeta function

Definition

The **Riemann-zeta function** is the function of the complex variable \( s \), defined in the half-plane of \( \mathbb{C} \) with real part of \( s \) greater than 1 by the absolutely convergent series

\[
\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}
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and in the whole complex plane \( \mathbb{C} \) by analytic continuation.
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- From Calculus II, what values $s$ were you taught that $\zeta(s)$ converges?
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- What does analytic continuation mean? [Student HW]
Enter Leonard Euler (April 15, 1707 – September 18, 1783)

A result of Euler connects the infinite sum \( \sum \) with an infinite product of infinite sums over the primes. The essence of Euler's proof is his use of the fundamental theorem of arithmetic to observe that the sum \( \sum \) can be written as the following infinite product:

\[
\prod_{n \geq 1} \left( 1 + \frac{1}{p_1^{s_n}} \right) \left( 1 + \frac{1}{p_2^{s_n}} \right) \left( 1 + \frac{1}{p_3^{s_n}} \right) \cdots
\]
A result of Euler connects the infinite sum $\zeta(s)$ with an infinite product of infinite sums over the primes. The essence of Euler’s proof is his use of the fundamental theorem of arithmetic to observe that the sum $\zeta(s)$ can be written as the following infinite product

$$\sum_{n \geq 1} \frac{1}{n^s} = \prod_p \left( 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \ldots \right)$$
To prove Equation (1), Euler observed the following:

By the fundamental theorem of arithmetic each \( n \) in the denominator on the left-hand side is of the form 
\[ n = p_{i_1}^{i_1} p_{i_2}^{i_2} \cdots p_{i_k}^{i_k} \]
for some \( k \).

Then by multiplying out the product on the right-hand side, each term \( \frac{1}{n^s} \) on the left-hand side appears exactly once, as a product of the appropriate powers of the primes in \( n \).

And since each multiplicand on the right-hand side is a geometric series of the form 
\[ \frac{1}{1 - \frac{1}{p^s}} \],
Equation (1) becomes

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\sum_{n \geq 1} \frac{1}{n^s} = \prod_p \left( 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \cdots \right)
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\sum_{n \geq 1} \frac{1}{n^s} = \prod_p \left( 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \cdots \right) \tag{1}
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2. And since each multiplicand on the right-hand side is a **geometric series** of the form \( \frac{1}{1 - \frac{1}{p^s}} \), Equation (1) becomes

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\sum_{n \geq 1} \frac{1}{n^s} = \prod_p \frac{1}{1 - \frac{1}{p^s}}.
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Some values of $\zeta(s)$ for positive integer $s$ values

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<tr>
<td>4</td>
<td>$\pi^4/90$</td>
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For the even positive integers, the values of the zeta function are

$$\zeta(2n) = (-1)^{n+1} \frac{B_{2n}(2\pi)^{2n}}{2(2n)!}$$

for $n \in \mathbb{N}$, and the $B_{2n}$ is the $2n^{th}$ Bernoulli number.
Some values of $\zeta(s)$ for positive integer $s$ values

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For the even positive integers, the values of the zeta function are

$$\zeta(2n) = (-1)^{n+1} \frac{B_{2n} (2\pi)^{2n}}{2(2n)!}$$

for $n \in \mathbb{N}$, and the $B_{2n}$ is the $2n^{th}$ Bernoulli number. What is the definition of a Bernoulli number? [Student HW]
Outline

1. Introduction
2. What is the Riemann hypothesis?
3. What is the Riemann-zeta function?
4. The apparent “convergence” of some divergent series?
5. Analytic continuation of the zeta function
6. Visualizing the zeros of the zeta function
Does $\zeta(-1) = \sum_{n=1}^{\infty} \frac{1}{n^{-1}} = 1 + \frac{1}{2^{-1}} + \frac{1}{3^{-1}} + \frac{1}{4^{-1}} + \cdots$ converge?
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3. Does the series $\zeta(-2) = 1 + 2^2 + 3^2 + 4^2 + \cdots$ converge?

4. Would you believe that $\zeta(-1) = -\frac{1}{12}$ and that $\zeta(-2) = 0$?
Who believes that $1 + 2 + 3 + 4 + \cdots = -\frac{1}{12}$?

ANSWER:

Physicists (in particular string theorists) believe!

And also physicists involved in QED (Quantum Electrodynamics) believe!

FACT: Riemann's analytic continuation of $(s)$ to $\mathbb{C}$ yields $(−1)^s = −\frac{1}{12}$.

Some genius mathematicians who derived that $1 + 2 + 3 + 4 + \cdots = −\frac{1}{12}$ without resorting to the analytic continuation of $(s)$?

Leonard Euler

Srinivasa Ramanujan
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- Leonard Euler
- Srinivasa Ramanujan
“Dear Sir, I am very much gratified on perusing your letter of the 8th February 1913. I was expecting a reply from you similar to the one which a Mathematics Professor at London wrote asking me to study carefully Bromwich’s *Infinite Series* and not fall into the pitfalls of divergent series… I told him that the sum of an infinite number of terms of the series: $1 + 2 + 3 + 4 + \cdots = -\frac{1}{12}$ under my theory. If I tell you this you will at once point out to me the lunatic asylum as my goal…”
In Chapter 8 of his first notebook, Ramanujan presented two derivations of 1 + 2 + 3 + 4 + ⋯ = −\frac{1}{12}. The simpler, less rigorous derivation proceeds as follows.

Let \( c = 1 + 2 + 3 + 4 + 5 + 6 + \cdots \).

Then we get

The alternating series 1−2+3−4+⋯ is the formal power series expansion of the function \( \frac{1}{1+1} \) but with \( x \) defined as 1.

Ramanujan concludes that

\[ -3c = 1−2+3−4+\cdots = \frac{1}{4} \]

Finally, dividing both sides by −3 we get

\( c = -\frac{1}{12} \).

Let's discuss problems with this argument!
In Chapter 8 of his first notebook, Ramanujan presented two derivations of $1 + 2 + 3 + 4 + \cdots = -\frac{1}{12}$. The simpler, less rigorous derivation proceeds as follows. Let $c = 1 + 2 + 3 + 4 + 5 + 6 + \cdots$. 

The alternating series $1 - 2 + 3 - 4 + \cdots$ is the formal power series expansion of the function $f(x) = \frac{1}{(1+x)^2}$ but with $x$ defined as $1$. Ramanujan concludes that $-3c = 1 - 2 + 3 - 4 + \cdots = \frac{1}{4}$.

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    c &= 1 + 2 + 3 + 4 + 5 + 6 + \cdots \\
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What did Riemann do?

Riemann defined the unique analytic continuation of \( \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \) extending the domain of \( \zeta(s) \) to \( \mathbb{C} \) as a meromorphic function with only a simple pole at \( s = 1 \) with residue 1.
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\pi^{-\frac{s}{2}} \Gamma \left( \frac{s}{2} \right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma \left( \frac{1-s}{2} \right) \zeta(1-s),
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[Student HW]
Some values of $\zeta(s)$ for negative integer $s$ values

Recall the values of $\zeta(s)$ for the following positive $s$ values.

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In general for integers $n > 0$, we have $\zeta(-n) = (-1)^n\frac{B_{n+1}}{n+1}$ where $B_{n+1}$ is the $(n + 1)^{th}$ Bernoulli number.
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In general for integers $n > 0$, we have $\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1}$ where $B_{n+1}$ is the $(n + 1)^{th}$ Bernoulli number. What is $B_m$ when $m$ is odd? [Student HW]
How do we extend the domain of \( \zeta(s) \) beyond \( \Re(e(s)) > 1 \)?

Let us rename our zeta function as \( Z(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \).
How do we extend the domain of $\zeta(s)$ beyond $\Re(s) > 1$?

Let us rename our zeta function as $Z(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$.

It is known that $Z$ is an absolutely convergent series for all values $s \in \mathbb{C}$ whenever $\Re(s) > 1$. 

Goal #1: Extend the domain of $Z$ to all $s \in \mathbb{C}$ whenever $\Re(s) > 0$.

Multiplying both sides of Equation (2) by $1 - 2^{2s}$, we get the following observing that the sum on the right is now alternating.
How do we extend the domain of \( \zeta(s) \) beyond \( \Re(s) > 1 \)?

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\[
Z(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots
\]  

(2)
How do we extend the domain of $\zeta(s)$ beyond $\Re(s) > 1$?

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$$Z(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots \quad (2)$$

Multiplying both sides of Equation (2) by $\left(1 - \frac{2}{2^s}\right)$, we get the following

$$\left(1 - \frac{2}{2^s}\right)Z(s) = \frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \cdots \quad [\text{Student HW}]$$

observing that the sum on the right is now alternating.
Dividing both sides of the previous equation by \(1 - \frac{2}{2^s}\), we get

\[
Z(s) = \frac{1}{1^s - 2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \cdots
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(3)

Hence Goal #1 is accomplished, and we have extended \(\zeta(s)\) to \(\Re(s) > 0\). Let us denote this extended domain function as \(\zeta\).
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\[
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**Goal #2:** Extend the domain of \(\zeta\) to all \(s \in \mathbb{C}\) whenever \(\Re(s) < 0\).
Dividing both sides of the previous equation by \(1 - \frac{2}{2^s}\), we get

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**Goal #2:** Extend the domain of \(\zeta\) to all \(s \in \mathbb{C}\) whenever \(\Re(s) < 0\).

We utilize the functional equation in the following form:

\[
\zeta(s) = 2^s \pi^{s-1} \sin \left( \frac{\pi s}{2} \right) \Gamma(1 - s) \zeta(1 - s).
\]
Dividing both sides of the previous equation by \( 1 - \frac{2}{2^s} \), we get

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This will allow us to evaluate \( \zeta \) for values of \( s \) with \( \Re(s) < 0 \).
A more rigorous evaluation of $\zeta(-1) = 1 + 2 + 3 + 4 + \cdots$

Recall the previous functional equation

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$$\zeta(s) = 2^s \pi^{s-1} \sin \left( \frac{\pi s}{2} \right) \Gamma(1 - s) \zeta(1 - s).$$

Setting $s = -1$, we get

$$\zeta(-1) = 2^{-1} \pi^{-1-1} \sin \left( \frac{-\pi}{2} \right) \Gamma(1 - (-1)) \zeta(1 - (-1)).$$

So what do we conclude?

Exercise: Compute $\zeta(0)$, $\zeta(-2)$, and $\zeta(-3)$. [Student HW]
A more rigorous evaluation of $\zeta(-1) = 1 + 2 + 3 + 4 + \cdots$

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$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1 - s) \zeta(1 - s).$$

Setting $s = -1$, we get

$$\zeta(-1) = 2^{-1} \pi^{-1-1} \sin\left(\frac{-\pi}{2}\right) \Gamma(1 - (-1)) \zeta(1 - (-1)).$$

Recall that $\zeta(2) = \frac{\pi^2}{6}$. And since $\Gamma(n) = (n - 1)!$ then we get

$$\zeta(-1) = \frac{1}{2} \pi^{-2} \sin\left(\frac{-\pi}{2}\right) 1! \frac{\pi^2}{6}.$$
A more rigorous evaluation of \( \zeta(-1) = 1 + 2 + 3 + 4 + \cdots \)

Recall the previous functional equation

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So what do we conclude?
A more rigorous evaluation of $\zeta(-1) = 1 + 2 + 3 + 4 + \ldots$

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So what do we conclude?

**Excercise:** Compute $\zeta(0)$, $\zeta(-2)$, and $\zeta(-3)$. [Student HW]
Outline

1. Introduction
2. What is the Riemann hypothesis?
3. What is the Riemann-zeta function?
4. The apparent “convergence” of some divergent series?
5. Analytic continuation of the zeta function
6. Visualizing the zeros of the zeta function
Visualizing the Riemann-zeta function

As a complex valued function of a complex variable, the graph of the Riemann zeta function $\zeta(s)$ lives in four dimensional real space. To get an idea of what the function looks like, we must do something clever.
Visualizing the Riemann-zeta function

As a complex valued function of a complex variable, the graph of the Riemann zeta function $\zeta(s)$ lives in four dimensional real space. To get an idea of what the function looks like, we must do something clever.

Level curves to the rescue!!

The real and imaginary parts of $\zeta(s)$ are each real valued functions; we can think of the graphs of each one as a surface in three dimensional space. Rather than look at two surfaces simultaneously, we can view the level curves for the two surfaces. The level curves are curves in the $s$-plane showing points of constant height on the surface, as on a contour map.
Level curves for $\Re e\left(\frac{1}{s}\right)$ are shown above with the solid lines; the red curve is $\Re e\left(\frac{1}{s}\right)=0$, the black curves represent values other than zero.

Level curves for $\Im m\left(\frac{1}{s}\right)$ are shown above with dotted lines; the green curve is $\Im m\left(\frac{1}{s}\right)=0$, the black curves represent values other than zero.
Level curves for $\Re(\zeta(s))$ are shown above with the solid lines; the red curve is $\Re(\zeta(s)) = 0$, the black curves represent values other than zero.
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• Level curves for $\Im(\zeta(s))$ are shown above with dotted lines; the green curve is $\Im(\zeta(s)) = 0$, the black curves represent values other than zero.
The function \( f(s) \) is real on the real axis, thus \( \Im\left(f(s)\right) = 0 \) there.

Zeros of \( f(s) \) are points in the plane where both \( \Re\left(f(s)\right) = 0 \) and \( \Im\left(f(s)\right) = 0 \); these are points where the red and green curves cross.

You can see the trivial zeros at the negative even integers, and the first nontrivial zero at \( s = \frac{1}{2} + i \cdot 14.13 \ldots \), this is the point in the plane \((\frac{1}{2}, 14.13 \ldots)\). You can also see the pole at \( s = 1 \).
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2. Zeros of $\zeta(s)$ are points in the plane where both $\text{Re}(\zeta(s)) = 0$ and $\text{Im}(\zeta(s)) = 0$; these are points where the red and green curves cross.
3. You can see the trivial zeros at the negative even integers, and the first nontrivial zero at $s = \frac{1}{2} + i \cdot 14.135 \ldots$, this is the point in the plane $(\frac{1}{2}, 14.135 \ldots)$. You can also see the pole at $s = 1$. 
Your first homework

1. I will pair you up with a partner and you will work together on this first assignment.

2. There are nine reasonable homework problems in these slides. These are noted with the symbol [Student HW].

3. Be prepared to present the homework solution for your group on our next class meeting.

Meanwhile Riemann says

My beard is better at math than you.