**The Fibonacci and Lucas Sequences**

**Definition:** The **Fibonacci sequence** is the sequence \((F_n)_{n \geq 0}\) with the recurrence relation \(F_{n-1} + F_n = F_{n+1}\) and two initial conditions \(F_0 = 0\) and \(F_1 = 1\).

Fill in the missing blanks [You Do!]

<table>
<thead>
<tr>
<th>(F_0)</th>
<th>(F_1)</th>
<th>(F_2)</th>
<th>(F_3)</th>
<th>(F_4)</th>
<th>(F_5)</th>
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<th>(F_8)</th>
<th>(F_9)</th>
<th>(F_{10})</th>
<th>(F_{11})</th>
<th>(F_{12})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td></td>
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</tr>
</tbody>
</table>

**Definition:** The **Lucas sequence** is the sequence \((L_n)_{n \geq 0}\) with the recurrence relation \(L_{n-1} + L_n = L_{n+1}\) and two initial conditions \(L_0 = 2\) and \(L_1 = 1\).

Fill in the missing blanks [You Do!]

<table>
<thead>
<tr>
<th>(L_0)</th>
<th>(L_1)</th>
<th>(L_2)</th>
<th>(L_3)</th>
<th>(L_4)</th>
<th>(L_5)</th>
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</tr>
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<tbody>
<tr>
<td>2</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Do you see any connection between these two lists of numbers?  
(e.g., add two Fibonacci numbers that are two apart like \(F_4 + F_6\))
FIBONACCI AND LUCAS IDENTITIES

Exercise 1: Add two Fibonacci numbers that are two apart. [You Do!]

<table>
<thead>
<tr>
<th></th>
<th>F₀</th>
<th>F₁</th>
<th>F₂</th>
<th>F₃</th>
<th>F₄</th>
<th>F₅</th>
<th>F₆</th>
<th>F₇</th>
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<th>F₉</th>
<th>F₁₀</th>
<th>F₁₁</th>
<th>F₁₂</th>
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</thead>
<tbody>
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<td>21</td>
<td>34</td>
<td>55</td>
<td>89</td>
<td>144</td>
</tr>
</tbody>
</table>

Exercise 2: Add Fibonacci and Lucas numbers in same column. [You Do!]

<table>
<thead>
<tr>
<th></th>
<th>L₀</th>
<th>L₁</th>
<th>L₂</th>
<th>L₃</th>
<th>L₄</th>
<th>L₅</th>
<th>L₆</th>
<th>L₇</th>
<th>L₈</th>
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<th>L₁₀</th>
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</tr>
</thead>
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<tr>
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<td>11</td>
<td>18</td>
<td>29</td>
<td>47</td>
<td>76</td>
<td>123</td>
<td>199</td>
<td>322</td>
</tr>
</tbody>
</table>

Conjectured Formula: [You Do!]

Conjectured Formula: [You Do!]
**Exercise 3:** Add two Lucas numbers that are two apart. [You Do!]

<table>
<thead>
<tr>
<th>F₀</th>
<th>F₁</th>
<th>F₂</th>
<th>F₃</th>
<th>F₄</th>
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Conjectured Formula: [You Do!]

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**Exercise 4:** Multiply Fibonacci and Lucas numbers in same column.

[You Do!]

<table>
<thead>
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<th>L₀</th>
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Conjectured Formula: [You Do!]
**FIBONACCI AND LUCAS IDENTITIES (continued)**

**Exercise 5:** Take the difference of two Fibonacci numbers that are four apart (e.g., \(F_7 - F_3\)). [You Do!]

<table>
<thead>
<tr>
<th>(F_0)</th>
<th>(F_1)</th>
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**Conjectured Formula:** [You Do!]

\[
L_0 \quad L_1 \quad L_2 \quad L_3 \quad L_4 \quad L_5 \quad L_6 \quad L_7 \quad L_8 \quad L_9 \quad L_{10} \quad L_{11} \quad L_{12}
\]
| 2      | 1      | 3      | 4      | 7      | 11     | 18     | 29     | 47     | 76     | 123    | 199    | 322    |

**Exercise 6:** Conjecture a formula for \(F_{n+1}L_{n+1} - F_nL_n\). [You Do!]

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The Beautiful Binet Formula 😊

I am Jacques Philippe Marie Binet. I was born in 1786 and died in 1856. The Binet formula is named after me in 1843, but it was known over 100 years earlier by Abraham de Moivre in 1718. But my last name is easier to pronounce. Ha.

**EXERCISE 7:** What is the 100th Fibonacci number? Can I compute it without knowing the 98th and 99th Fibonacci numbers? [You Do!]

**Hint:** Binet’s formula states that

\[ F_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right] \]

**EXERCISE 8:** What is the 100th Lucas number? Can I compute it without knowing the 98th and 99th Lucas numbers? [You Do!]

**Hint:** A “Binet”-ish formula states that

\[ L_n = \left( \frac{1 + \sqrt{5}}{2} \right)^n + \left( \frac{1 - \sqrt{5}}{2} \right)^n \]
The Golden Ratio

**EXERCISE 9:** Where do those two numbers $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$ come from?

**Hint:** Use the quadratic formula to find the roots of $x^2 - x - 1 = 0$.

What does this have to do with beauty?

**EXERCISE 10:** Let us denote phi (Greek letter $\Phi$) to be this golden. Then $\Phi = \frac{A}{B}$. Then from the equality $\frac{A}{B} = \frac{A+B}{B}$, we can derive $\Phi = 1 + \frac{1}{\Phi}$. Multiply this across by $\Phi$ and what do you get? [You Do!]

![Diagram of the golden ratio]

\[ \frac{A}{B} = \frac{A+B}{B} = 1.618 = \Phi \]
The Golden Ratio (continued)

**QUESTION:** Does the golden ratio appear everywhere in nature and man?

Here is a picture of some random guy whose hair and side of face model the golden spiral:

Here is a famous image of the Mona Lisa painting by ________________(?)
**The Golden Ratio (continued)**

**QUESTION:** How does the golden ratio and the Fibonacci numbers arise in sunflowers? [You Do!]

**EXERCISE 11:** Calculate $\frac{F_{n+1}}{F_n}$ for $n = 9, 10, \text{ and } 11$. What do you notice about these ratios? Use the Binet formula to calculate $\frac{F_{51}}{F_{50}}$. [You Do!]
The Stair-Climbing Problem

**QUESTION:** How many ways to climb up **two** stairs if you can only take one step or two steps at a time?

**ANSWER:** [You Draw!]

**QUESTION:** How many ways to climb up **three** stairs if you can only take one step or two steps at a time?

**ANSWER:** [You Draw!]

**QUESTION:** How many ways to climb up **four** stairs if you can only take one step or two steps at a time?

**ANSWER:** Here is a picture of the answer.

![Five ways to climb four stairs](image)

The answer is 5 ways, since we can do the following five step sequences:

1111, 211, 121, 112, and 22.
EXERCISE 12: How many ways to climb up \( n \) stairs if you can only take one step or two steps at a time?

**ANSWER:** Let \( y_n \) be the number of ways to climb up \( n \) stairs if you can only take one or two steps at a time. For example, we learned on the previous page that \( y_2 = \_\_\_\_\_\_\_ \), \( y_3 = \_\_\_\_\_\_\_ \), and \( y_4 = \_\_\_\_\_\_\_ \). So, find a closed form for \( y_n \). [You Do!]

EXERCISE 13: How many ways to climb up 49 stairs if you can only take one step or two steps at a time? [You Do!]
The Generalized Fibonacci Sequence

**EXERCISE 14:** Suppose we start with two initial conditions $G_0 = a$ and $G_1 = b$ and we have the same “Fibonacci/Lucas” recurrence relation $G_{n-1} + G_n = G_{n+1}$ for all $n \geq 1$. Prove that we have the following closed form $G_n = F_{n-1}G_0 + F_nG_1$. [You Do!]

**EXERCISE 15:** Suppose $G_0 = 3$ and $G_1 = 1$. Use the recurrence relation $G_{n-1} + G_n = G_{n+1}$ to compute $G_2, G_3, G_4, G_5,$ and $G_6$. Verify that $G_6$ equals 23 by using the closed form $G_n = F_{n-1}G_0 + F_nG_1$. [You Do!]

**EXERCISE 16:** Suppose $G_0 = 7$ and $G_1 = -2$. Use the recurrence relation $G_{n-1} + G_n = G_{n+1}$ to compute $G_2, G_3, G_4, G_5,$ and $G_6$. Verify that $G_6$ equals 19 by using the closed form $G_n = F_{n-1}G_0 + F_nG_1$. [You Do!]
The Fibonacci Matrix

Definition: The Fibonacci matrix is the following $2 \times 2$ matrix

$$Q = \begin{pmatrix} F_2 & F_1 \\ F_1 & F_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$ 

EXERCISE 17: Do matrix multiplication and compute $Q^2$, $Q^3$, $Q^4$, and $Q^5$. What pattern do you notice? Conjecture what matrix you would get if you had $Q^n$. [You Do!]

EXERCISE 18: Compute the determinant (i.e., phattie Dee) for $Q$, $Q^2$, $Q^3$, $Q^4$, and $Q^5$. What pattern do you notice? Conjecture what the determinant of $Q^n$ is. [You Do!]
Pascal’s Triangle “hides” the Fibonacci Sequence

**EXERCISE 19:** Can you find the Fibonacci sequence “hidden” inside Pascal’s triangle below? [You Do!]

Hint: Consider some sums of Pascal’s triangle entries.

**EXERCISE 20:** Consider the formula below where \( j = \left\lfloor \frac{n-1}{2} \right\rfloor \) is the largest integer not exceeding \( \frac{n-1}{2} \). For example, if \( n = 8 \), then \( \left\lfloor \frac{8-1}{2} \right\rfloor = 3 \) since \( \frac{7}{2} = 3 \frac{1}{2} \). Verify the formula below for \( n = 8 \). [You Do!]

\[
F_n = \binom{n-1}{0} + \binom{n-2}{1} + \binom{n-3}{2} + \cdots + \binom{n-j}{j-1} + \binom{n-j-1}{j}.
\]
EXERCISE 21: Consider the 4\textsuperscript{th} row of Pascal’s triangle. Multiply each number left-to-right by the Fibonacci numbers $F_0, F_1, F_2$, etc. Then add the numbers in that row. What do you notice? [You Do!] Do the same for the 5\textsuperscript{th} row. What do you notice? [You Do!]

EXERCISE 22: Conjecture a closed form for the following sum in the $n^{th}$ row of Pascal’s triangle: [You Do!]

$$\binom{n}{0} F_0 + \binom{n}{1} F_1 + \binom{n}{2} F_2 + \cdots + \binom{n}{n-1} F_{n-1} + \binom{n}{n} F_n.$$
Back Page History

Who is Fibonacci?

- He lived from around 1170 to 1240. He was born in the Republic of Pisa (currently Italy).
- His name is NOT Fibonacci. The term is short for the Latin “filius Bonacci” (i.e., the son of the Bonnacio’s; his father’s name was Guglielmo Bonnacio).

Who is Édouard Lucas?

- He lived from 1842 to 1891. He was born in France.
- He died in unusual circumstances. At a banquet, a waiter dropped some crockery and a piece of broken plate cut Lucas on the cheek. He died a few days later of a severe skin inflammation probably caused by sepsis. He was only 49 years old. 😞
Is This Page Intentionally Left Blank for Your Doodling Delight?  

Nope. It is placed here to ensure that the new section on the next page begins with an odd-numbered page.