On a coin-flip problem and its connection to π

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On February 18, 2011, Donald Knuth delivered the Christie Lecture at Bowdoin College. His talk was centered around a very simple yet penetrating geometric proof given by Johan Wästlund in 2007 on Wallis' product formula for π. In his talk, Knuth made very interesting additions to Wästlund's short note [1]. It is my goal to give a synthesis of both Wästlund's and Knuth's ideas. To do so, I incorporate both Wästlund's original ideas and some ideas that Knuth stated in his 16th Annual Christmas Tree Lecture delivered at Stanford University. The latter talk was a slightly different version of the Bowdoin talk of the same title *Why Pi?*, and hence if you were at the Bowdoin talk then this write-up will include more than was given there. Also I was keeping students in mind when writing this, so many of the cool and interesting steps in the proofs are left as exercises.

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1 Introduction to the flipping coins problem

Suppose we flip a fair coin $2n$ times. What are the chances that we get an equal amount of heads and tails? It turns out that the answer to this question involves π. Label the head and tail sides of the coin as H and T respectively. We can view the $2n$ coin flips as sequences of H's and T's. We can then rephrase the original question to the following: Out of the 2^{2n} possible sequences, how many sequences contain exactly n H's and n T's?

Example 1.1. Consider the case for $n = 5$. Then the number of ways of getting exactly five H's is the number of ways of choosing 5 of the 10 sequence slots in which to place them. Hence there are

$$\binom{10}{5} = \frac{10!}{5!\ \ 5!} = 252$$

ways. Now since there are exactly $2^{10} = 1024$ possible sequences of H's and T's, the probability of getting exactly five H's and five T's is $\frac{252}{1024} = .24609375$. This number is close to $\frac{1}{\sqrt{5\pi}}$ which is approximately .252313. For $n = 20$, the probability is .125371 while $\frac{1}{\sqrt{20\pi}}$ is approximately .126157.

Claim 1.2. *As n gets large, the probability of getting exactly n heads and n tails out of $2n$ coin flips approaches the number* $\dfrac{1}{\sqrt{n\pi}}$.

2 Drunken walks

Consider a walk where we go forward each time however at each step forward we veer 45 degrees left or right. The question of the coin flip problem can be recast in this drunken walk model if we choose to let H, for instance, be a step forward that veers left and T to be one that veers right. Hence if after $2n$ steps we return to the original horizontal starting level, then we must have taken exactly n left-steps and n right-steps. Consider the number of drunken walks of four steps that return to the original horizontal starting level. There are six possibilities as shown in Figure 1.

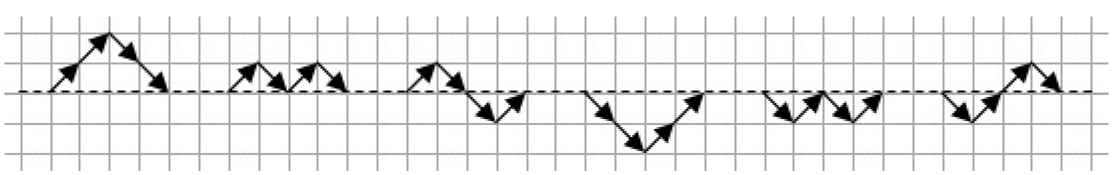

Figure 1: M_2 walks

Now consider the number of drunken walks of four steps that never return to the original horizontal starting level. It is no coincidence that this yields six possibilities also, as shown in Figure 2.

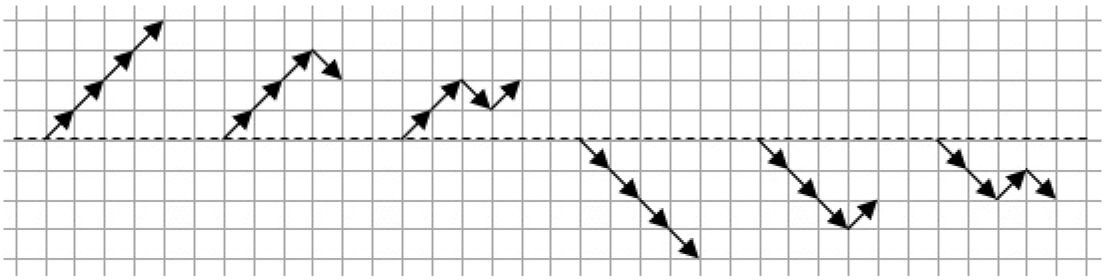

Figure 2: N_2 walks

Definition 2.1. Let M_n be the number of cases of $2n$-walks that return to the original horizontal starting level. And let N_n be the number of cases of $2n$-walks that never revisit the original horizontal starting level after initially leaving it.

The probability of getting exactly n heads and n tails in $2n$ coin flips is $\frac{M_n}{2^{2n}}$. Furthermore, the following remarkable identity holds:

$$2^{2n} = M_n N_0 + M_{n-1} N_1 + \cdots + M_1 N_{n-1} + M_0 N_n. \tag{1}$$

We list the first few M_n and N_n values in the table below.

n	M_n	N_n
0	1	1
1	2	2
2	6	6
3	20	20
4	70	70
5	252	252

Does M_n equal N_n for all $n \geq 0$? This and many other fun exercises are presented in the next section.

Remark 2.2. A cool way to see that Identity (1) holds is to view it as a partition of all possible drunken walks on $2n$ steps. Consider an arbitrary walk. In any such walk there will be a portion of the walk of the form M_i for some $i \leq n$, followed by a walk that is of the form N_j for $j = n - i$. By abuse of language and notation we use the term *walk of the form*

M_i *(respectively* N_j*)* to mean a walk that leaves and returns to the initial starting level after $2i$ steps (respectively, a walk that leaves the initial starting level and never returns again for the remaining $2j$ steps). Observe that it might happen that we cross the horizontal starting point several times before the end, but there will certainly be a LAST time that we cross it – at the $2i^{th}$ step. Afterwards, we are either above or below the horizontal starting level for the remaining $2j$ steps. For instance, for six drunken steps (the $n = 3$ case), each walk of the possible $2^6 = 64$ walks will lie in exactly **one** of the M_3N_0, M_2N_1, M_1N_2, or M_0N_3. Since $M_2 = 6$ and $N_1 = 2$, there are exactly 12 drunken walks of the form M_2N_1. We give two such walks in the figure below.

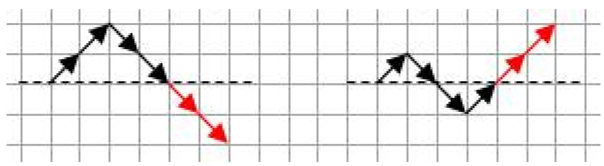

Figure 3: Two of the 12 possible M_2N_1 walks

3 Exercises left for the reader

Exercise 3.1. *Let* $n \geq 0$ *be given. Then* M_n *equals* N_n*.*

Exercise 3.2. *Let* a_n *denote the quotient* $\frac{M_n}{2^{2n}}$*. Then* a_n *is our desired probability. Using Exercise 3.1 and Identity (1), it follows that the following identity holds*

$$1 = a_n a_0 + a_{n-1} a_1 + \cdots + a_1 a_{n-1} + a_0 a_n.$$

Definition 3.3. Define s_n to be the sum $a_0 + a_1 + \cdots + a_{n-1}$. Define $s_0 = 0$. The following table gives the first five a_n and s_n values.

n	a_n	s_n
0	1	0
1	$\frac{1}{2}$	1
2	$\frac{3}{8}$	$\frac{3}{2}$
3	$\frac{5}{16}$	$\frac{15}{8}$
4	$\frac{35}{128}$	$\frac{35}{16}$

The next exercise shows that the probabilities a_n and their sums s_n can be written in a rather remarkable way.

Exercise 3.4. *For $n \geq 1$, each a_n can be written as the product of quotients $\frac{1}{2}\frac{3}{4}\cdots\frac{2n-1}{2n}$. Consequently for $n > 1$, each s_n can be written as the product $\frac{3}{2}\frac{5}{4}\cdots\frac{2n-1}{2n-2}$.*

Hint: For the a_n-case rewrite $\frac{1}{2}\frac{3}{4}\cdots\frac{2n-1}{2n}$ as $\frac{1}{2}\frac{2}{2}\cdot\frac{3}{4}\frac{4}{4}\cdots\frac{2n-1}{2n}\frac{2n}{2n}$ and show this equals $\dfrac{M_n}{2^{2n}}$.

Exercise 3.5. *From the previous exercise, it follows that $s_{n+1} - s_n = a_n$.*

Exercise 3.6. *For each n, it follows that $s_n = 2n \cdot a_n$.*

Definition 3.7. Let r be a real number and let k be an integer. We define the binomial coefficient $\binom{r}{k}$ by

$$\binom{r}{k} = \begin{cases} \dfrac{r(r-1)\cdots(r-k+1)}{k!} & \text{if } k \geq 1 \\ 1 & \text{if } k = 0 \\ 0 & \text{if } k \leq -1. \end{cases}$$

For example,

$$\binom{-\frac{1}{2}}{3} = \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!} = \frac{-5}{16}.$$

Exercise 3.8 (Alternate way to express a_n)**.** *The value a_n is equal to the following value:*

$$a_n = \binom{-\frac{1}{2}}{n}(-1)^n.$$

The previous exercise gives a useful expression that helps generalize the a_n to other settings as introduced in Section 8 on superellipses.

4 Where does π come into the picture?

As said in Claim 1.2, the probability of getting exactly n heads and n tails in $2n$ coin flips involves π. Specifically, we intend to show that the probability a_n asymptotically approaches the value $\dfrac{1}{\sqrt{n\pi}}$. We show this by using nothing more than the Pythagorean theorem, the area formula for a circle, elementary algebra, and a simple geometric series sum. The following Figure 4 motivates our reasoning.

The area of the sums of all the rectangles is 5. This is easily seen from Exercise 3.2, which tells us that the following sums of products of the form $a_n a_0 + a_{n-1}a_1 + \cdots + a_1 a_{n-1} + a_0 a_n$

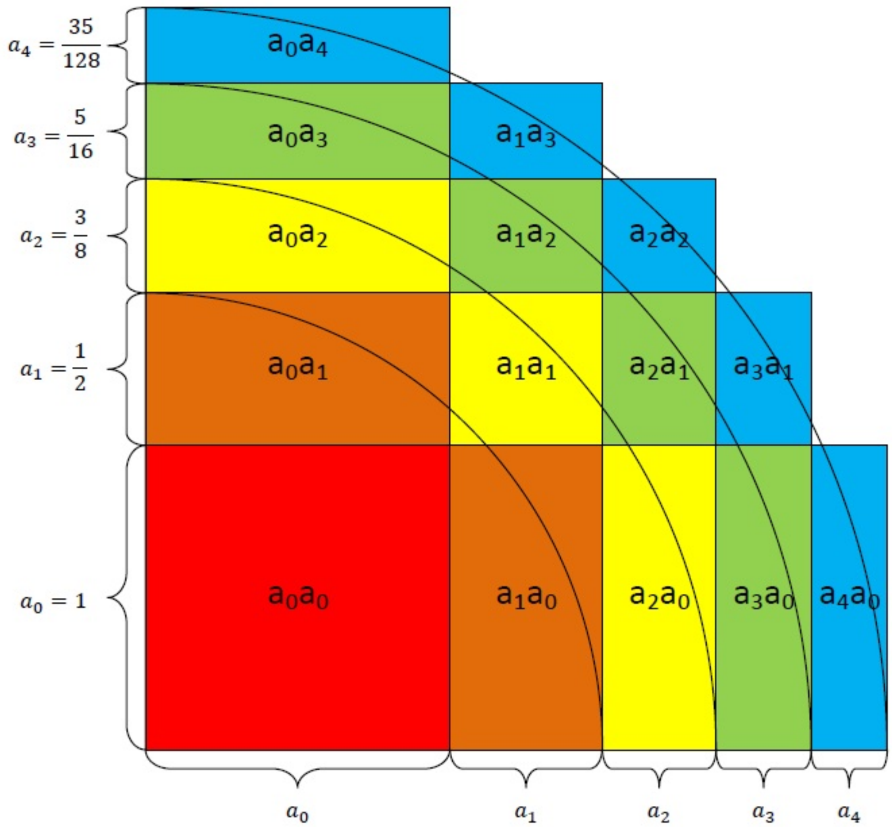

Figure 4: Sums of areas of rectangles

each equals one. It follows that

$$
\begin{aligned}
\text{Red rectangle area} &= a_0a_0 = 1 \\
\text{Orange rectangle areas} &= a_1a_0 + a_0a_1 = 1 \\
\text{Yellow rectangle areas} &= a_2a_0 + a_1a_1 + a_0a_2 = 1 \\
\text{Green rectangle areas} &= a_3a_0 + a_2a_1 + a_1a_2 + a_0a_3 = 1 \\
\text{Blue rectangle areas} &= a_4a_0 + a_3a_1 + a_2a_2 + a_1a_3 + a_0a_4 = 1.
\end{aligned}
$$

Moreover, the rectangles arranged in this fashion begin to resemble a quarter circle with radius $s_5 = a_0 + a_1 + \cdots + a_4$ in the case of Figure 4. Hence the area $\frac{1}{4}\pi s_5^2$ is approximately equal to 5. Precisely, since $s_5 = \frac{315}{128}$ the value $\frac{1}{4}\pi s_5^2$ equals 4.75654. As we add more rectangles, the outermost rectangles appear more and more to sit exactly on the circumference of the quarter circle. That is, the value $\frac{1}{4}\pi s_n^2$ is approximately n as n gets large. If we can show this, then we have essentially proven Claim 1.2. This follows since

$$n \approx \frac{1}{4}\pi(s_n)^2 = \frac{1}{4}\pi(2na_n)^2 = \pi n^2 a_n^2,$$

where the second equality follows by Exercise 3.6. Hence $a_n^2 \approx \dfrac{1}{n\pi}$ and thus a_n approaches $\dfrac{1}{\sqrt{\pi n}}$ as n gets large. More precisely, in the next section we prove that $n-1 < \frac{1}{4}\pi s_n^2 < n+1$. This implies that $\frac{1}{4}\pi s_n^2 = n+O(1)$. Hence $s_n^2 = \frac{4}{\pi}n+O(1)$, which we rewrite as $\frac{4}{\pi}n(1+O(\frac{1}{n}))$. Thus $s_n = \sqrt{\frac{4n}{\pi}(1+O(\frac{1}{n}))}$, implying the result

$$a_n = \frac{s_n}{2n} = \frac{1}{\sqrt{\pi n}} + O\left(\frac{1}{n^{3/2}}\right).$$

5 A simple geometric proof

The heart of the proof that $a_n \longrightarrow \dfrac{1}{\sqrt{n\pi}}$ is a clever geometric insight by Wästlund in his proof of Wallis' product formula [1]. At each stage n, the right sides of the outermost rectangles (as in Figure 4) lie on either side of the circumferences of the quarter circles with radius s_n. Furthermore as n gets larger, these rectangles get thinner and thinner and begin to squeeze the circumference. The following Figure 5 motivates our reasoning.

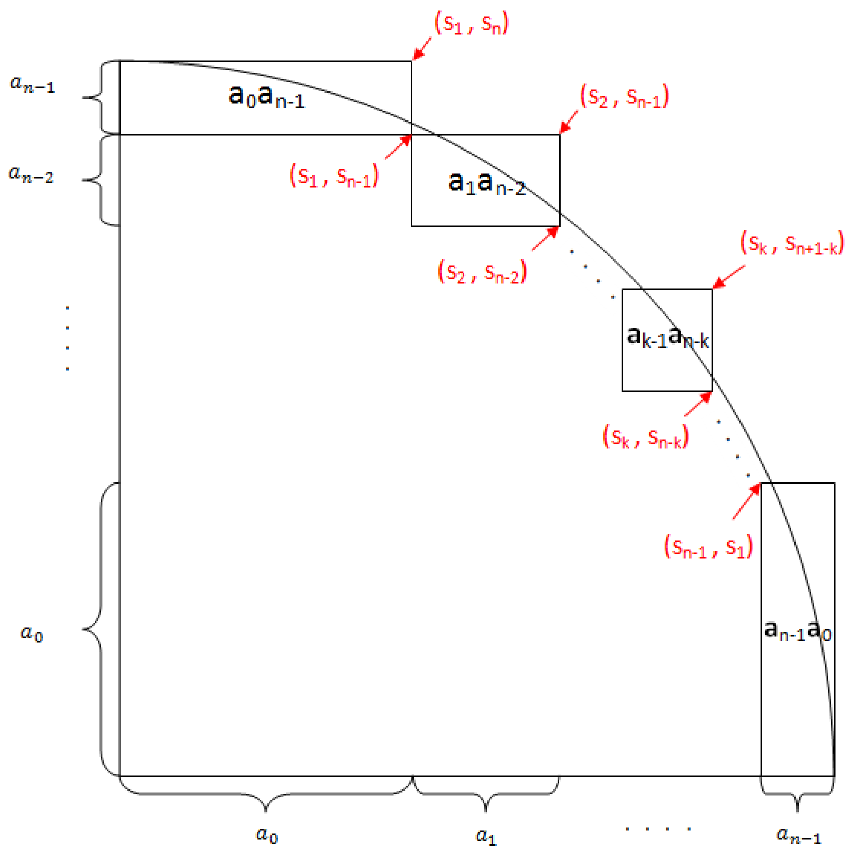

Figure 5: Geometric motivation

Goal 5.1. *Show that* $n - 1 < \frac{1}{4}\pi s_n^2 < n + 1$.

It suffices to show that for each n, that the bottom right corner (s_k, s_{n-k}) of each outermost rectangle is left of the circumference while the top right corner (s_k, s_{n+1-k}) is right of the circumference. To do this we use the Pythagorean theorem in the following guise.

Claim 5.2. *Let* $k, n \in \mathbb{N}$ *such that* $1 \leq k \leq n-1$. *The distance from the origin to the inner notches* (s_k, s_{n-k}) *is less than* s_n.

Claim 5.3. *Let* $k, n \in \mathbb{N}$ *such that* $2 \leq k \leq n$. *The distance from the origin to the outer notches* (s_k, s_{n+1-k}) *is greater than* s_n.

The proofs of the two claims follow easily once we are armed with two simple strict inequalities involving s_i^2 and s_j^2 for arbitrary $i \neq j$ (which we leave as Exercises 6.1 and 6.2 in the next section). We also leave the proofs of these two claims above as Exercise 6.3. The inequality exercises are manageable to prove once the following motivating example is understood.

Example 5.4. Consider the values $s_3 = \frac{15}{8}$ and $s_5 = \frac{315}{128}$ written in their product of quotients representations from Exercise 3.4. Then $s_3 = \frac{3}{2}\frac{5}{4}$ and $s_5 = \frac{3}{2}\frac{5}{4}\frac{7}{6}\frac{9}{8}$. Squaring both expressions, we get

$$s_3^2 = \left(\frac{3}{2}\right)^2\left(\frac{5}{4}\right)^2 \quad \text{and} \quad s_5^2 = \left(\frac{3}{2}\right)^2\left(\frac{5}{4}\right)^2\left(\frac{7}{6}\right)^2\left(\frac{9}{8}\right)^2.$$

Thus $s_5^2 = s_3^2 \cdot \left(\frac{7}{6}\right)^2\left(\frac{9}{8}\right)^2$. Notice that $\left(\frac{7}{6}\right)^2$ and $\left(\frac{9}{8}\right)^2$ can be written in the form $\left(1 + \frac{1}{2k}\right)^2$ for $k = 3$ and $k = 4$ respectively. But $\left(1 + \frac{1}{2k}\right)^2 = 1 + \frac{1}{k} + \frac{1}{4k^2} > 1 + \frac{1}{k}$. Thus we conclude

$$s_5^2 = s_3^2 \cdot \left(\frac{7}{6}\right)^2\left(\frac{9}{8}\right)^2 > s_3^2 \cdot \left(1 + \frac{1}{3}\right)\left(1 + \frac{1}{4}\right) = s_3^2 \cdot \frac{4}{3} \cdot \frac{5}{4} = s_3^2 \cdot \frac{5}{3}.$$

On the other hand, using the fact that $\left(1 + \frac{1}{2k}\right)^2 < \frac{k}{k-1}$ (an argument proven in Lemma 5.5 using geometric series), we conclude

$$s_5^2 = s_3^2 \cdot \left(\frac{7}{6}\right)^2\left(\frac{9}{8}\right)^2 < s_3^2 \cdot \frac{3}{2} \cdot \frac{4}{3} = s_3^2 \cdot \frac{4}{2}.$$

Lemma 5.5. *For* $k \geq 1$ *it follows that* $\left(1 + \dfrac{1}{2k}\right)^2 < \dfrac{k}{k-1}$.

Proof. Observe that $\frac{k}{k-1}$ can be written as $\dfrac{1}{1 - \frac{1}{k}}$ and hence it has geometric series representation as $1 + \frac{1}{k} + \frac{1}{k^2} + \cdots$. Thus it suffices to show that

$$\left(1 + \frac{1}{2k}\right)^2 < 1 + \frac{1}{k} + \frac{1}{k^2} + \cdots.$$

For $k = 1$ this follows, so consider $k \geq 2$. Since $4k > k - 1$, we know $4k^2 > k(k-1)$ and hence $\frac{1}{4k^2} < \frac{1}{k(k-1)}$. Then we conclude

$$\begin{aligned}
\left(1 + \frac{1}{2k}\right)^2 = 1 + \frac{1}{k} + \frac{1}{4k^2} &< 1 + \frac{1}{k} + \frac{1}{k(k-1)} \\
&= 1 + \frac{1}{k} + \frac{k}{k^2(k-1)} \\
&= 1 + \frac{1}{k} + \frac{1}{k^2}\left(\frac{1}{1 - \frac{1}{k}}\right) \\
&= 1 + \frac{1}{k} + \frac{1}{k^2} + \frac{1}{k^3} + \cdots
\end{aligned}$$

as desired. □

Theorem 5.6 (Wallis, 1655)**.** *The following product formula holds:*

$$\prod_{n=1}^{\infty} \left(\frac{2n}{2n-1} \cdot \frac{2n}{2n+1}\right) = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{8}{7} \cdot \frac{8}{9} \cdots = \frac{\pi}{2}.$$

Proof. Consider Euler's formula for the sine function using the following infinite product:

$$\frac{\sin(x)}{x} = \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right).$$

If we let $x = \dfrac{\pi}{2}$, then we get the following series of implications:

$$\begin{aligned}
\frac{\sin(x)}{x} = \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right) &\Rightarrow \frac{2}{\pi} = \prod_{n=1}^{\infty} \left(1 - \frac{1}{4n^2}\right) \\
&\Rightarrow \frac{\pi}{2} = \prod_{n=1}^{\infty} \left(\frac{4n^2}{4n^2 - 1}\right) \\
&\Rightarrow \frac{\pi}{2} = \prod_{n=1}^{\infty} \left(\frac{2n}{2n-1} \cdot \frac{2n}{2n+1}\right) \\
&\Rightarrow \frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{8}{7} \cdot \frac{8}{9} \cdots,
\end{aligned}$$

as desired. □

Exercise 5.7. *The proof of Wallis' product formula for π given above is not the standard textbook proof. Usually, most textbook proofs rely on an evaluation of a definite integral like*

$$\int_0^{\frac{\pi}{2}} (\sin x)^n \, \mathrm{d}x$$

by repeated partial integration. Try to prove Wallis' product formula using this method.

6 More fun exercises

Exercise 6.1. *Prove that* $s_j^2 < s_i^2 \cdot \dfrac{j-1}{i-1}$ *for* $j > i \geq 2$.

Exercise 6.2. *Prove that* $s_j^2 > s_i^2 \cdot \dfrac{j}{i}$ *for* $j > i \geq 1$.

Exercise 6.3. *Use the previous two exercises to prove Claims 5.2 and 5.3*

Hint: To prove the first claim, notice that $s_n^2 > s_k^2 \cdot \frac{n}{k}$ and $s_n^2 > s_{n-k}^2 \cdot \frac{n}{n-k}$. Use that to bound $s_k^2 + s_{n-k}^2$ above by a multiple of s_n^2.

7 Wästlund's approach

Wästlund gives an elementary algebraic way to derive the identity

$$1 = a_n a_0 + a_{n-1}a_1 + \cdots + a_1 a_{n-1} + a_0 a_n.$$

His original paper does not speak of coin tosses or drunken walk probabilities. He merely defines the simple product of quotients $s_n = \frac{3}{2}\frac{5}{4}\cdots\frac{2n-1}{2n-2}$ for $n \geq 2$ and $s_1 = 1$. And then he defines a_n to be the difference $s_{n+1} - s_n$, which has the equally simple representation as $\frac{1}{2}\frac{3}{4}\cdots\frac{2n-1}{2n}$.

Exercise 7.1. *The following identity holds*

$$a_i a_j = \frac{j+1}{i+j-1} a_i a_{j+1} + \frac{i+1}{i+j-1} a_{i+1} a_j.$$

Exercise 7.2. *Observe that* $a_0^2 = 1$. *By repeated applications of the identity in the previous exercise, we get the following string of equalities:*

$$\begin{aligned} 1 = a_0^2 &= a_1 a_0 + a_0 a_1 \\ &= a_2 a_0 + a_1 a_1 + a_0 a_2 \\ &= \quad \vdots \\ &= a_n a_0 + a_{n-1} a_1 + \cdots + a_1 a_{n-1} + a_0 a_n. \end{aligned}$$

8 A generalization and avenues for students to study

The circle $x^2 + y^2 = r^2$ generalizes to superellipses via the equation $x^{\frac{1}{\alpha}} + y^{\frac{1}{\alpha}} = r^{\frac{1}{\alpha}}$. So when $\alpha = \frac{1}{2}$ we get the circle.

Question 8.1. *What happens when $\alpha = \frac{1}{3}$. Compute some a_n values and draw a picture similar to the colorful rainbow Figure 4? Can we generalize Wästlund's argument to this new setting?*

Our first few a_n values are $a_0 = 1$, $a_1 = \frac{1}{3}$, $a_2 = \frac{1}{3}\frac{4}{6}$, $a_3 = \frac{1}{3}\frac{4}{6}\frac{7}{9}$. See the pattern? It is a good exercise (like Exercise 3.8) to show that this and other nice patterns appear if we prove $a_n = \binom{-\alpha}{n}(-1)^n$ in general. Then in our case, let $\alpha = \frac{1}{3}$ to see the desired pattern above.

I have not had time to verify any of the following claims (and hence there may be a typo somewhere below) but all of the following would be good for students to explore.

- For $\alpha = \frac{1}{3}$, those a_n values seem to have a nice representation as a product of quotients. Do the s_n have a nice representation too (as they did in the $\alpha = \frac{1}{2}$ case)?
- $s_n = \binom{-1-\alpha}{n}(-1)^n$
- $s_n = \frac{n}{\alpha} \cdot a_n$
- Let $b_n = a_n a_0 + a_{n-1}a_1 + \cdots + a_1 a_{n-1} + a_0 a_n$. In the $\alpha = \frac{1}{2}$ case, we saw that each b_n equals 1. Is there an equally nice formula for b_n for arbitrary α.
- The area of the superellipse $x^{\frac{1}{\alpha}} + y^{\frac{1}{\alpha}} = 1$ is
$$\frac{2\alpha(\Gamma(\alpha))^2}{\Gamma(2\alpha)}.$$
- In the $\alpha = \frac{1}{2}$ case, we saw that $a_n \longrightarrow \dfrac{1}{\sqrt{n\pi}}$. For $\alpha = \frac{1}{3}$, does a_n approach something just as cool?

9 Acknowledgments

Aba Mbirika thanks Rosemary Roberts for conversing with him about the M_n and N_n diagrams and clearing up some ideas after the talk. These conversations prompted him to think deeper about Don Knuth's very interesting talk and the work of Johan Wästlund. Coincidentally, Aba TEX'ed most of this up on March 14 which is Pi Day!

References

[1] Wästlund, J.: An elementary proof of the Wallis product formula for pi. *Amer. Math. Monthly* **14**(no.10), 914–917 (2007)