RESEARCH STATEMENT

Aba Mbirika

1. Introduction

Combinatorial representation theory sits at a nexus of combinatorics, representation theory, topology, commutative algebra, and algebraic geometry. Representation theory can be viewed as a machine which inputs an algebra $\mathfrak{A}$ and outputs information about $\mathfrak{A}$-modules (that is, the representations of $\mathfrak{A}$). Combinatorial representation theorists are especially interested in using counting arguments to study its smallest unbreakable parts, namely the *irreducible* $\mathfrak{A}$*-modules*. They count them, compute their characters and dimensions, and try to construct them. The projects I am interested in span a wide array of combinatorial analysis. The four I will speak about in this statement are the following:

- **Project I:** Cohomology ring of nilpotent Hessenberg varieties
- **Project II:** Sign representations for imprimitive complex reflection groups
- **Project III:** A Schur-Weyl duality for the alternating group
- **Project IV:** Prime labeling of graphs

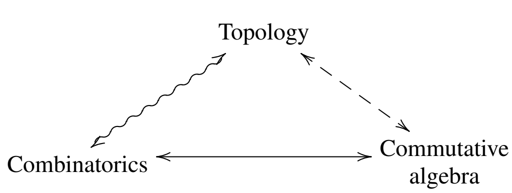
Projects I and II are the primary research projects that I have been working on during my postdoctoral position here at Bowdoin College. Project I involves giving a combinatorial description of a certain cohomology ring. In a particular setting this ring can be used to construct irreducible representations of the symmetric group. Project II involves computing the sign representations of imprimitive complex reflection groups via a set of combinatorial objects associated to elements of these groups.

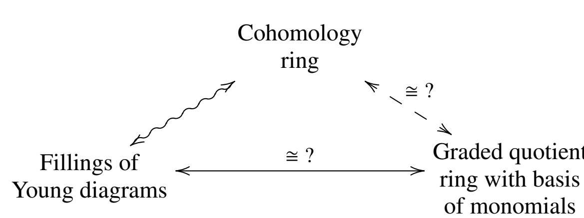
The next two projects, Projects III and IV, lend themselves well to undergraduate involvement. Project III is more classical in nature and involves Schur-Weyl duality, which also produces irreducible representations. Project IV is a graph theory project in which we study prime labelings of certain graphs. This project arose out of my recent residency at an AIM-NSF sponsored workshop at the ICERM institute in Rhode Island. My research collaborators and I plan to continue this work possibly at the AIM center in Palo Alto, California, in 2013. We also plan to involve undergraduates in future projects related to this topic.

It would be impossible to give a deep overview of all four projects without omitting many important details. So instead I will only give a detailed overview of Project I in which my previous two published papers were focused. Afterwards, I will provide a brief overview of Projects II, III, and IV.

2. Cohomology ring of nilpotent Hessenberg varieties

PROJECT I: Cohomology ring of nilpotent Hessenberg varieties. I explore a generalization of the Springer variety called Hessenberg varieties $\mathfrak{H}(X,h)$. Springer varieties are a family of varieties whose cohomology ring carries an important representation of the symmetric group and hence has been well studied [16, 3, 7, 18]. Much less is known about the cohomology of Hessenberg varieties. However in 2005, Tymoczko [19] offered a first glimpse (see Equation 1) – a combinatorial method to count the dimension of each graded part of this ring $H^*(\mathfrak{H}(X,h))$ using objects called Young tableaux. I describe a map (see Equation 2) from these Young tableaux to a set of monomials. In the Springer case this map extends to a graded vector space isomorphism and the monomials correspond exactly to a cohomology basis identified by Garsia-Procesi (see Subsection 2.1). The question is, "Can this map be generalized to non-Springer Hessenberg varieties, and in this new setting are the monomials still meaningful?" The answer is yes for the class of regular nilpotent Hessenberg varieties (see Subsection 2.2). The following schematics show how I weave the various branches of math in this research.





University of Wisconsin–Eau Claire http://people.uwec.edu/mbirika mbirika@uwec.edu

Background and motivation. As early as 1900, Alfred Young gave a combinatorially-driven construction of irreducible symmetric group modules. More recently in 1978, T.A. Springer gave a deep and geometric construction using cohomology of a family of varieties [16]. Fourteen years later, Garsia and Procesi made Springer's work more transparent and gave a presentation of this cohomology ring as a graded quotient of a polynomial ring [7]. My research generalizes Garsia-Procesi's work to the Hessenberg variety setting.

In 1992, De Mari, Procesi, and Shayman introduced Hessenberg varieties [4]. They play an important role in areas of numerical analysis, geometric representation theory, algebraic geometry, and are used in an efficient implementation of the QR-algorithm for matrix eigenvalue problems. Define a *Hessenberg function* h as a map from $\{1, 2, \ldots, n\} \longrightarrow \{1, 2, \ldots, n\}$ subject to the constraint that $i \leq h(i) \leq h(i+1)$. We denote this function as an n-tuple $h = (h_1, \ldots, h_n)$ where $h_i = h(i)$. A *flag* is a nested sequence of $\mathbb{C}$-vector spaces $V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = \mathbb{C}^n$ where each V_i has dimension i. The collection of all such flags is called the *full flag variety*, $\mathfrak{F}$. Fix a nilpotent operator $X \in \mathrm{Mat}_n(\mathbb{C})$. We define our *Hessenberg variety* to be the following subvariety of the full flag variety:

$$\mathfrak{H}(X, h) = \{\text{Flags} \in \mathfrak{F} \mid X \cdot V_i \subseteq V_{h(i)} \text{ for all } i\}.$$

Since conjugates of X produce a variety homeomorphic to $\mathfrak{H}(X, h)$, it suffices to fix X in its Jordan canonical form. We choose to fix our nilpotent matrix with Jordan blocks of non-decreasing sizes $\mu_1 \geqslant \cdots \geqslant \mu_s > 0$ so that $\sum_{i=1}^{s} \mu_i = n$ which we may view as a partition μ of n, or as a *Young diagram* μ with row lengths μ_i. For a fixed X, there are two extreme cases. If $h = (1, 2, \ldots, n)$, then $\mathfrak{H}(X, h)$ is the *Springer variety*. If $h = (n, \ldots, n)$, then $\mathfrak{H}(X, h) = \mathfrak{F}$. We will always take h to be in this range $(1, 2, \ldots, n) \leq h \leq (n, \ldots, n)$.

In 2005, Tymoczko gave a combinatorial procedure for finding the dimensions of the graded parts of the cohomology ring $H^*(\mathfrak{H}(X, h))$ [19]. Let the Young diagram μ correspond to the Jordan form of a nilpotent operator $X \in \mathrm{Mat}_n(\mathbb{C})$. Let T be a filling of the n boxes of μ with the numbers $1, 2, \ldots, n$ injectively. If T satisfies certain combinatorial rules, it is called an *(h, μ)-filling* and corresponds to a basis element of $H^*(\mathfrak{H}(X, h))$. Let a *dimension-pair* be a particular kind of inversion described in [19]. Tymoczko proves:

$$dim(H^{2k}(\mathfrak{H}(X, h))) = \text{ the number of } (h, \mu)\text{-fillings } T \text{ such that } T \text{ has } k \text{ dimension pairs.} \tag{1}$$

The dimension pairs come in tuples (x, y) such that $1 \leq x < y \leq n$. The total number of dimension pairs possessed by a filling T is said to be the *dimension of T*. Denote DP_y^T to be the set of pairs of the form $(\cdot, y)$. I defined a graded morphism from these (h, μ)-fillings onto a set of monomials:

$$\Phi : (h, \mu)\text{-fillings} \longmapsto \mathcal{A}_h(\mu) \quad \text{defined by} \quad T \longmapsto \prod_{\substack{(i,j) \in \mathrm{DP}_j^T \\ 2 \leq j \leq n}} x_j \tag{2}$$

where the notation $\mathcal{A}_h(\mu)$ is used to denote the image of this map. By abuse of notation we denote the linear span of these monomials by the same symbol. Extending Φ linearly, we get a map on vector spaces.

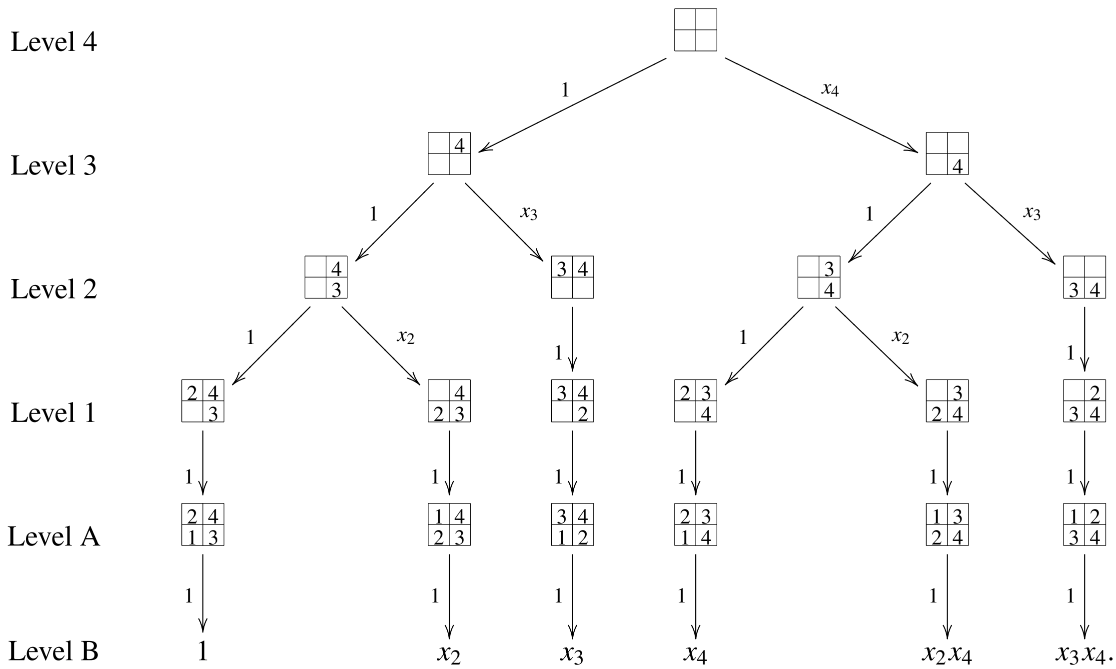
Theorem 2.1 (Mbirika)**.** *If μ is a partition of n, then Φ is a well defined degree-preserving map from (h, μ)-fillings to monomials in $\mathcal{A}_h(\mu)$. Namely, r-dimensional (h, μ)-fillings map to r-degree monomials in $\mathcal{A}_h(\mu)$.*

2.1. **My results in the Springer setting.** In the Springer setting the set of (h, μ)-fillings are exactly the row-strict fillings of μ, which are the basis of tabloids for the complex vector space M^μ. Since h is fixed in this setting, we denote our monomials $\mathcal{A}_h(\mu)$ simply as $\mathcal{A}(\mu)$. And as in the literature, we will denote the Garsia-Procesi basis of the cohomology ring of the Springer variety as $\mathcal{B}(\mu)$.

Theorem 2.2 (A map from $\mathcal{A}(\mu)$ back to Tabloids)**.** *If μ is a partition of n, then there exists a well defined dimension-preserving map Ψ from monomials in $\mathcal{A}(\mu)$ to tabloids in M^μ. That is, r-degree polynomials in $\mathcal{A}(\mu)$ map to r-dimensional tabloids in M^μ. Moreover, $\mathcal{A}(\mu) \xrightarrow{\Psi} M^\mu \xrightarrow{\Phi} \mathcal{A}(\mu)$ is the identity.*

Corollary 2.3. *($\mathcal{A}(\mu)$ and M^μ are naturally bijective)*
$M^\mu \xrightarrow{\Phi} \mathcal{A}(\mu) \xrightarrow{\Psi} M^\mu$ is the identity. And hence $\mathcal{A}(\mu)$ and M^μ are isomorphic as graded vector spaces.

It turns out that our set of monomials $\mathcal{A}(\mu)$ coincides with the Garsia-Procesi basis $\mathcal{B}(\mu)$. Garsia and Procesi used a tree on Young diagrams to find $\mathcal{B}(\mu)$. We refined their construction to build a *tableau tree for* μ. The product of the edge labels at Level B gives the Garsia-Procesi basis $\mathcal{B}(\mu)$. Moreover, each path leading to a particular monomial in $\mathcal{B}(\mu)$ gives the exact prescription to building the filling (at Level A) which indeed is an (h,μ)-filling and will map under Φ to the monomial below it. That gives our inverse map Ψ. Here is an example of the $\mu = (2,2)$ case:



Theorem 2.4 (Mbirika)**.** *Let μ be a Young diagram and consider its corresponding tableau tree. Each of the fillings at Level A are (h,μ)-fillings. Moreover, they map under Φ to the corresponding $\mathbf{x}^\alpha \in \mathcal{B}(\mu)$ at the end of this path. Since Level A also gives all possible (h,μ)-fillings, we have $\mathcal{A}(\mu) = \mathcal{B}(\mu)$.*

2.2. **My results in the regular nilpotent Hessenberg setting.** What if we now let h vary? Are these new monomials still meaningful? If we change the h-function, Ψ no longer maps reliably back to the original filling. For example if $h = (1,3,3)$ then $\Phi\left(\begin{smallmatrix}3&2\\1&\end{smallmatrix}\right) = x_3$, but $\Psi(x_3) = \begin{smallmatrix}1&2\\3&\end{smallmatrix}$. Attempts so far to define an inverse map that work for all h-functions and all shapes μ have been unsuccessful.

However, when we fix the shape $\mu = (n) = \square\square\cdots\square$ and let the h-function vary, we get a very important family of varieties called the *regular nilpotent Hessenberg varieties*, $\mathfrak{H}(X^{reg}, h)$. In this setting, the image of Φ is again a meaningful set of monomials $\mathcal{A}_h(\mu)$. They coincide with a basis $\mathcal{B}_h(\mu)$ of a particular polynomial quotient ring R/J_h that we model after the Garsia-Procesi quotient. However, R is now the polynomial ring $\mathbb{Z}[x_1, \ldots, x_n]$, and J_h is an ideal generated by truncated complete symmetric functions. In my first paper, I conjectured that $H^*(\mathfrak{H}(X^{reg}, h))$ is isomorphic to the polynomial quotient ring R/J_h, identified the basis of monomials $\mathcal{B}_h(\mu)$, and proved that this basis recovers the Betti numbers of $\mathfrak{H}(X^{reg}, h)$ [9].

2.3. **Evidence in the Peterson variety subclass.** The family of regular nilpotent Hessenberg varieties contains a small subclass of varieties called *Peterson varieties*, Y. These are the $\mathfrak{H}(X^{reg}, h)$ for which the Hessenberg function is defined as $h(i) = i + 1$ for $i < n$ and $h(n) = n$. These objects are a central figure in the study of the quantum cohomology of the flag variety. Harada and Tymoczko recently gave the first general computation of $H^*(Y)$ in terms of generators and relations [8]. Their presentation is given via a Monk-type formula. Although computable, the presentation is computationally heavy. Software such as Macaulay 2 is needed to produce small examples and exhibit a basis (via Gröbner basis reduction). We have evidence (in small n cases) that $H^*(Y)$ and R/J_h, for h as defined in this subsection, are ring isomorphic. Advantages of my conjectural presentation are three-fold:

(1) The ideal J_h is easily computed without computer software.
(2) Since its generators form a Gröbner basis already, computing the basis of R/J_h is a simple task.
(3) My presentation generalizes to all regular nilpotent Hessenbergs (i.e., for all h).

2.4. **More evidence in my second paper and generalized Tanisaki ideals I_h.** Tymoczko and I define a new family of ideals I_h, which generalize the Tanisaki ideals (denoted $\mathcal{I}_X$), in my second paper [10]. To do this, we modified methods from Biagioli-Faridi-Rosa's work on finding reduced generating sets for the $\mathcal{I}_X$ ideals [1]. We prove that this new presentation I_h coincides with J_h. Moreover, upon reading a preprint of our paper, a Jerzy Weyman student Federico Galetto proved that our conjectural minimal generating sets for I_h is indeed minimal. Interestingly enough, it was Weyman himself who first conjectured a minimal generating set for $\mathcal{I}_X$ in 1989 [20] to which Biagioli-Faridi-Rosas gave a counterexample [1]. However, their methods helped us develop techniques to find minimal generating sets for our I_h ideals. Perhaps a combination of the two methods can be used to settle the still open problem of finding minimal generating sets for the $\mathcal{I}_X$ ideals. After hearing a talk I gave on our generalized Tanisaki ideals, Aaron Lauve of Loyola University Chicago (who independently has been thinking of minimalized generating sets for the classic Tanisaki ideals) expressed interest in pursuing this project with me.

2.5. **Ongoing work with Tymoczko and a conjecture of Reiner.** Tymoczko and I are exploring a connection of my monomials $\mathcal{B}_h(\mu)$ to a collection of permutations that actually are fixed points of a union of Schubert varieties. This comes from a (personally communicated) conjecture of Vic Reiner that the ring $\mathbb{Z}[x_1, \ldots, x_n]/J_h$ which I identified is actually the cohomology of a family of Schubert varieties (very similar to Ding varieties) [5]. This fixed point result would help analyze the relationship between the cohomology rings of certain Schubert varieties and certain Hessenberg varieties. In the near future, we intend to see how this work relates to Church-Ellenberg-Farb's theory of representation stability [2]. Lastly, other ideas for future work (including undergraduates where applicable) include the following:

- Extend the inverse map Ψ_h to all shapes, not just the regular nilpotent $\mu = (n)$ case.
- The symmetric group acts on the cohomology ring of the Springer variety. This action is more accessible when viewed as an action on the quotient ring presentation where $\sigma \in \mathfrak{S}_n$ acts on subscripts of the monomials. Is there an action on the arbitrary Hessenberg cohomology rings?

3. Brief overview and goals of Projects II, III, and IV

PROJECT II: Sign representations for imprimitive complex reflection groups. The *Robinson-Schensted algorithm* (RS) gives a bijective correspondence between the symmetric group $\mathfrak{S}_\mathrm{n}$ and pairs of same-shape standard Young tableaux. This correspondence is known to relate certain permutation statistics with certain tableaux statistics. For example, Schensted showed that the length of the longest increasing (resp., decreasing) subsequence in a permutation equals the length of the first row (resp., column) of the corresponding tableaux. In particular, Reifegerste (and independently, Sjöstrand [14]) showed how the sign of a permutation can be deduced from its tableaux image [13]. For a tableau T, we call a pair (i, j) of entries an *inversion* of T if $j < i$ and j is contained in a row below the row of i. Let $\mathrm{inv}(T)$ denote the set of inversions of T. Defining $\mathrm{sign}(T) = (-1)^{|\,\mathrm{inv}(T)|}$ to be the *sign* of tableau T, she proves the following.

Theorem 3.1 (Reifegerste [13])**.** *Let τ be a permutation in $\mathfrak{S}_\mathrm{n}$, and let $(P(\tau), Q(\tau))$ be its corresponding image under the RS-map. Then* $\mathrm{sgn}(\tau) = (-1)^e \cdot \mathrm{sign}(P(\tau)) \cdot \mathrm{sign}(Q(\tau))$*, where e is the total length of the even-indexed rows of $P(\tau)$.*

Example 3.2. Consider the element $\tau = 2\,6\,1\,4\,5\,3 \in \mathfrak{S}_6$, whose sign of -1 can be computed directly by counting inversions of the permutation. Under the Robinson-Schensted correspondence, we have

$$RS(\tau) = (P(\tau), Q(\tau)) = \left(\begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2} 6\\\cline{1-1}\end{array}\, , \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4\\\cline{1-2} 6\\\cline{1-1}\end{array} \right).$$

The total length of the even-indexed rows of $P(\tau)$ is 2, $|\mathrm{inv}(P(\tau))| = 3$, $|\mathrm{inv}(Q(\tau))| = 2$, and hence Reifegerste's formula gives $\mathrm{sgn}(\tau) = (-1)^2 \cdot 1 \cdot -1 = -1$ as desired.

Following the work of Stanton and White, the RS-algorithm can be generalized to give a correspondence between the complex reflection groups $\mathbb{G}(\mathrm{r,p,n})$ and pairs of same shape multitableaux [17]. First we give a brief description of elements in $\mathbb{G}(\mathrm{r,p,n})$.

Definition 3.3. Let $r, p, n \in \mathbb{N}$ with $p|r$ and let $\zeta = \exp\left(\frac{2\pi\sqrt{-1}}{r}\right)$. The family of *imprimitive complex reflection groups* $\mathbb{G}(\mathrm{r,p,n})$ are the subgroups of $\mathrm{GL}_n(\mathbb{C})$ consisting of matrices such that

- the entries are either 0 or powers of ζ,
- there is exactly one nonzero entry in each row and column, and
- the (r/p)-th power of the product of all nonzero entries is 1.

For an element $w \in \mathbb{G}(r,p,n)$, we may use a *window notation* and write $w = [\zeta^{a_1}\sigma_1, \zeta^{a_2}\sigma_2, \ldots, \zeta^{a_n}\sigma_n]$ where $\sigma = \sigma_1 \cdots \sigma_n \in \mathfrak{S}_\mathrm{n}$. In this notation, w is the matrix whose nonzero entry in the i-th column is ζ^{a_i} and appears in row σ_i, and by the third condition above, $(\zeta^{a_1} \cdots \zeta^{a_n})^{\frac{r}{p}} = 1$ holds necessarily. Together with 34 exceptional groups, the groups $\mathbb{G}(\mathrm{r,p,n})$ account for all finite groups generated by complex reflections [15].

For more details on this definition and other combinatorial ways to view the groups $\mathbb{G}(\mathrm{r,p,n})$, a description of their one-dimensional irreducible representations, and the generalized RS-algorithm on them, consult my brief summary at:

http://people.uwec.edu/mbirika/Complex_reflection_groups_overview.pdf

Example 3.4. Consider $w = [\zeta^1 5, \zeta^0 1, \zeta^2 3, \zeta^0 6, \zeta^2 7, \zeta^1 4, \zeta^0 2] \in \mathbb{G}(4,1,7)$, where ζ is the primitive 4th root of unity. Under the generalized Robinson-Schensted correspondence, this element maps to a pair of same-shape multitableaux with 4 components each as follows:

$$\mathbf{RS}(w) = (\mathbf{P}(w), \mathbf{Q}(w)) = \left(\left(\begin{array}{|c|c|}\hline 1&2\\\hline 6\\\cline{1-1}\end{array}, \begin{array}{|c|}\hline 4\\\hline 5\\\hline\end{array}, \begin{array}{|c|c|}\hline 3&7\\\hline\end{array}, \emptyset \right), \left(\begin{array}{|c|c|}\hline 2&4\\\hline 7\\\cline{1-1}\end{array}, \begin{array}{|c|}\hline 1\\\hline 6\\\hline\end{array}, \begin{array}{|c|c|}\hline 3&5\\\hline\end{array}, \emptyset \right) \right).$$

A natural question to ask is whether Reifegerste's work can be replicated. Whereas the sign of a permutation in $\mathfrak{S}_\mathrm{n}$ gives 1 of 2 one-dimensional representations, the group $\mathbb{G}(\mathrm{r,p,n})$ has $2r$ one-dimensional irreducible representations of which r of them are sign representations $\mathrm{sgn}_i : \mathbb{G}(\mathrm{r,p,n}) \to \mathbb{C}$ for $0 \le i \le r$. Thomas Pietraho, Bill Silver and I have proven a formula that generalizes the Reifegerste result, and we show how all r sign representations of the group $\mathbb{G}(\mathrm{r,p,n})$ can be deduced from their multitableaux image. We submitted our joint paper in 2013 [12]. One important implication of our result is the following:

- In the symmetric and hyperoctahedral cases (i.e., when $r = 1, 2$, respectively), our result tells us how to read the Möbius function for the Bruhat lattice from the Robinson-Schensted image.

PROJECT III: A Schur-Weyl duality for the alternating group. A focal point of classical representation theory is Schur-Weyl duality. Let $V = \mathbb{C}^n$. Both the general linear group $\mathrm{GL}_n(\mathbb{C})$ and the symmetric group $\mathfrak{S}_k$ act on $V^{\otimes k}$, the k-fold tensor product of V. In 1927, Schur found that these two groups are mutual centralizers of each other, thus relating their representations. Subgroups of GL_n and their centralizers have been extensively studied. We list a few with their corresponding centralizers below each one.

$$\begin{array}{ccccccc} \mathrm{GL}_n(\mathbb{C}) & \supseteq & \mathrm{O}_n(\mathbb{C}) & \supseteq & \mathfrak{S}_n & \supseteq & \mathfrak{A}_n \\ \mathbb{C}\mathfrak{S}_k & \subseteq & \mathbb{C}B_k(n) & \subseteq & \mathbb{C}P_k(n) & \subseteq & \text{Unknown} \end{array}$$

The centralizer of GL_n can be viewed as the span of permutation diagrams. The centralizer of the orthogonal group O_n is the Brauer algebra $\mathbb{C}B_k(n)$, spanned by the permutation diagrams plus some extra diagrams. In 1993, the centralizer of the subgroup $\mathfrak{S}_n$ of permutation matrices sitting inside GL_n was presented as the partition algebra $\mathbb{C}P_k(n)$, a span of partition diagrams. The alternating group $\mathfrak{A}_n$ is the determinant 1 permutation matrices. Its centralizer has yet to be formulated. It should be a super-algebra of the partition algebra. I am attempting to describe this centralizer. For more information on the background and motivation for this problem, and my results thus far and future work, consult my brief summary at:

http://people.uwec.edu/mbirika/MSRI-result-Summer2009_draft.pdf

PROJECT IV: Prime labeling of graphs Graph labelings were first introduced in the late 1960s and have applications in areas spanning real-world settings to the purely theoretical. Dozens of graph labeling techniques have been studied in over 1000 papers since then [6]. One particular technique is called a *prime labeling*. A graph G with n vertices is called *prime* if we can bijectively label its vertices with the set $\{1, 2, \ldots, n\}$ so that the labels of any two adjacent vertices are relatively prime. We focus on two types of graphs: complete bipartite graphs and ladder graphs (see my expository paper [11] which includes a section with some recent results of my five collaborators and myself). Additionally, this research has some applications with which we feel we can involve undergraduates. This AIM-NSF sponsored project was started at the REUF4 (Research Experiences of Undergraduate Faculty 4) at ICERM in Summer 2012.

References

[1] R. Biagioli, S. Faridi, and M. Rosas, *The defining ideals of conjugacy classes of nilpotent matrices and a conjecture by Weyman*, International Mathematics Research Notices (2008).

[2] T. Church, J. Ellenberg, B. and Farb, *FI-modules: A new approach to stability for $\mathfrak{S}_n$-representations*. Preprint on arXiv, (2012). (Available at http://arxiv.org/pdf/1204.4533v2.pdf).

[3] C. De Concini and C. Procesi, *Symmetric functions, conjugacy classes, and the flag variety*, Invent. Math. **64** (1981), 203–230.

[4] F. De Mari, C. Procesi, and M. A. Shayman, *Hessenberg varieties*, Trans. Amer. Math. Soc. **332** (1992), 529–534.

[5] M. Develin, J. Martin, and V. Reiner, *Classification of Dings Schubert varieties: Finer rook equivalence*, Canad. J. Math. **59** No. 1 (2007), 36-62.

[6] J. Gallian, *A dynamic survey of graph labeling*, Electron. J. Combin. **(18)**, Dynamic Survey 6 (2011), 256pp.

[7] A. Garsia and C. Procesi, *On certain graded S_n-modules and the q-Kostka polynomials*, Adv. Math. **94** (1992), 82–138.

[8] M. Harada and J. Tymoczko, *A positive Monk formula in the S^1-equivariant cohomology of Type A Peterson varieties*, Proc. Lond. Math. Soc. (3) **103** No. 1 (2011), 40–72.

[9] A. Mbirika, *A Hessenberg generalization of the Garsia-Procesi basis for the cohomology ring of Springer varieties*, Electron. J. Combin. **17**(1), Research Paper 153 (2010).
(Available at http://www.combinatorics.org/Volume_17/PDF/v17i1r153.pdf).

[10] A. Mbirika and J. Tymoczko, *Generalizing Tanisaki's ideal via ideals of truncated symmetric functions*, Journal of Algebraic Combinatorics, **37** No. 1 (2013), 167–199.
(Arxiv version available at http://arxiv.org/pdf/1012.1630.pdf).

[11] A. Mbirika, *Hidden trees in the forest: On lattice points and prime labelings of graphs*, (primarily expository, but contains some unpublished results of myself and my collaborators A. Berliner, N. Dean, J. Hook, A. Marr, and C. McBee), (2012).
(Available at http://people.uwec.edu/mbirika/invisible_points_claim.pdf).

[12] A. Mbirika, T. Pietraho, and W. Silver, *On the sign representations for the complex reflection groups $G(r, p, n)$*.
(Submitted version available at http://people.uwec.edu/mbirika/paper_Elsevier_submit_version.pdf).

[13] A. Reifegerste, *Permutation sign under the Robinson-Schensted correspondence*, Annals of Combinatorics **8** No. 1 (2004), 103-112.

[14] J. Sjöstrand, *On the sign-imbalance of partition shapes*, J. Combin. Theory, Ser.A **111** (2005), 190–203.

[15] G. Shephard and J. Todd, *Finite unitary reflection groups*, Canadian J. Math. **6** (1954), 274-304.

[16] T. Springer, *A construction of representations of Weyl groups*, Invent. Math. **44** (1978), 279–293.

[17] D. Stanton and D. White, *A Schensted algorithm for rim hook tableaux*, J. Combin. Theory Ser. A **40** (1985), 211–247.

[18] T. Tanisaki, *Defining ideals of the closure of conjugacy classes and representations of the Weyl groups*, Tôhoku Math. J. **34** (1982), 575–585.
[19] J. Tymoczko, *Linear conditions imposed on flag varieties*, Amer. J. Math. **128** (2006), 1587–1604.
[20] J. Weyman, *The equations of conjugacy classes of nilpotent matrices*, Inventiones Mathematicae **98** No. 2 (1989), 229-45.