CONTAINMENT OF SUBGROUPS IN A DIRECT PRODUCT OF GROUPS

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DISSERTATION

Submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics in the Graduate School of Binghamton University State University of New York 2011
Accepted in partial fulfillment of the requirements for
the degree of Doctor of Philosophy in Mathematics
in the Graduate School of
Binghamton University
State University of New York
2011

April 14, 2011

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Abstract

Given a direct product of groups $G$ and $H$ and information on the subgroups of $G$ and $H$, what can one say about the subgroup structure of $G \times H$? In 1889, Edouard Goursat proved a theorem that allows us to obtain the subgroup structure of a direct product by examining the sections of the direct factors.

This dissertation provides a containment relation property between subgroups of a direct product. Namely, if $U_1, U_2 \leq G \times H$, we provide necessary and sufficient conditions for $U_2 \leq U_1$. We also show when two subgroups of a direct product are isomorphic. Using these results, we construct the subgroup lattices of two groups of order 64, $Q \times Q$ and $Q \times D_8$, where $Q$ and $D_8$ are the quaternion and dihedral groups of order 8 respectively. Lastly, we use these subgroup lattices to obtain and provide the subgroup lattices of the two extraspecial groups of order 32.
Acknowledgements

It is said that "the journey is more important than the destination." This has been an amazing journey full of many lessons learned and full of many people who have supported me, motivated me and who have helped me reach this destination.

To my advisor and friend, Ben Brewster, thank you for taking me on as your student, for taking a chance with me and for guiding me on this project. You have taught me so much in the past few years. I have learned how to do Group Theory, how to teach more effectively, how to write Mathematics better, how to give better presentations, and how to have fun doing and learning it all. You helped me see the bigger picture, and helped shape me into the Mathematician that I am today. Thank you for keeping me focused and for listening to all my ideas no matter how crazy some of them were!

To my committee members, Professors Fernando Guzmán and Marcin Mazur and Professor Joseph Petrillo of Alfred University, thank you for taking the time to serve on my committee, for taking interest in this project and for giving me suggestions. I would also like to thank Professor L.C. Kappe for taking the time to help prepare me for the job search and for carefully reading all my material.

To my family, thank you for always being there for me, for supporting me in my many adventures, for listening to my ideas, for motivating me when I felt like giving up and for believing in me when no one else did. To my mentors, thank you for exposing me and introducing me to the wonderful world of Mathematics and for giving me great advice. To my friends in the math department, thank you so much for helping me with latex!! To Q, thank you for being a great friend along this journey, and thank you for introducing me to Geometer’s Sketchpad. Without that, I do not know how these subgroup lattices would have made it in my dissertation. To all my other friends, thank you for all your help and for all the motivating conversations.
All of you have challenged, inspired and motivated me along this great journey, and now we are finally here. This is to new beginnings.
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Notation

Notation that is not cited here is consistent with that used in [2]. Functions are usually written on the left. However, in this dissertation there are times where we write them on the right as exponents. We trust that the reader will be able to decipher this.

\begin{itemize}
  \item $V_4$ the Klein 4-group
  \item $Q$ the quaternion group of order 8
  \item $D_8$ the dihedral group of order 8
  \item $S_n$ the symmetric group on $n$ letters
  \item $C_n$ the cyclic group of order $n$
  \item $Z(G)$ the center of $G$
  \item $\Phi(G)$ the Frattini subgroup of $G$
  \item $G'$ the commutator subgroup of $G$
  \item $Aut(G)$ the automorphism group of $G$
  \item $GF(p)$ the integers modulo $p$
  \item $Z_n$ the groups of order 2 in lattice of $Q \times Q$
  \item $\pi_G$ the natural projection of $G \times H$ onto $G$
  \item $\pi_H$ the natural projection of $G \times H$ onto $H$
  \item $|x|$ the order of a group element $x$
  \item $G \circdot H$ the central product of $G$ with $H$ [2]
  \item $G \bowtie H$ a subdirect product of $G$ and $H$ with amalgamated factor group [7] P. 50
\end{itemize}
Introduction

The main focus of this work is on a characterization of containment of subgroups in a direct product of finite groups. In particular, our primary goal is to establish this characterization and to give applications of this result. Suppose $G$ and $H$ are finite groups. Although subgroups of $G \times H$, their embeddings and their interactions with other subgroups have been investigated before (see [4], [6], [11]), finding properties which can further characterize the containment relation between subgroups remain open. This dissertation provides such a property.

Please note that throughout this dissertation, all of the groups discussed will be finite. Also note that our convention is that automorphisms will act on the right.

Our general approach to giving a characterization of containment of subgroups of a direct product and to providing applications of it is given in the outline to follow. Chapter 1 initiates this by providing background on the subgroup structure of a direct product and notation that will be used. In Section 1.1, we begin this discussion by presenting a theorem by Edouard Goursat, which will serve as a starting point for investigating containment of subgroups. Goursat’s Theorem provides a nice way of understanding the structure of a subgroup of a direct product. The reader should recall that the natural projections $\pi_A$ and $\pi_B$ are homomorphisms of $A \times B$ onto $A$ and $B$ respectively.

In Section 1.2, we expand on the notation introduced in Goursat’s Theorem, by presenting results that will provide a better understanding of mappings that will be used in our containment characterization. This section may seem very technical, and it is indeed. To follow the information presented here, the reader should recall the Isomorphism Theorems [3] and the projections and intersections that will be introduced in Section 1.1 after Goursat’s Theorem. To unify notation, we are explicit.

Section 1.3 concludes the chapter. This is the theoretical backbone of the dissertation.
There is a characterization of containment of subgroups of a direct product and a theorem that will allow one to see when two subgroups are isomorphic and have the same subgroup lattice structure. Suppose $U_1$ and $U_2$ are subgroups of $A \times B$. The characterization provides necessary and sufficient conditions for $U_2 \leq U_1$. The theorem that follows the characterization will then say when $U_1 \cong U_2$ under an automorphism of a direct product. We use $\text{Aut}(A)$ to denote the automorphism group of $A$.

More technically, the two results given in Section 1.3 are very applicable for those who study subgroup lattices because they provide us with another way of constructing subgroup lattices which have a large number of subgroups. In particular, using these results will allow us to construct two subgroup lattices of groups of order 64; namely, $Q \times Q$ and $Q \times D_8$, where $Q$ and $D_8$ denote the quaternion and dihedral groups of order 8.

In order to demonstrate this, we construct the subgroup lattice of $Q \times Q$. The details of this construction lie within Chapter 2. In Section 2.1, we use Goursat’s Theorem to calculate the number of subgroups in $Q \times Q$, and we provide a formula to calculate the order of the subgroups. We do this by first giving a presentation of $Q$ and the notation we will use for its subgroups. Then we use these subgroups of $Q$ and its subgroup lattice to examine the sections in each direct factor of $Q \times Q$. The reader should recall here that a section [2] of a group $A$ is just a quotient of a subgroup of the group $A$. After examining the sections of each factor we then calculate the orders of the groups.

In Sections 2.2 – 2.5, the details of the subgroups of $Q \times Q$ of orders 2 and 4, 8, 16 and 32 respectively will be given. The details of each section include providing notation for each of these subgroups, their group structures, and arguments which will give the maximal subgroups of each of these subgroups. Being that the factors of $Q \times Q$ are the same, one should expect symmetry among the groups and also expect that many subgroups will be isomorphic. Because we are in a 2-group, it is enough to determine the maximal subgroups from one level of the subgroup lattice to the next by order consideration.

In constructing the subgroup lattice, we will determine, by transitivity, only the maximal subgroups of the subgroups of $Q \times Q$. We lessen the amount of details involved by determining the maximal subgroups of one subgroup, say $U$, and then finding automorphisms which move the subgroup $U$ to another subgroup, say $W$. We then use this automorphism to determine the maximal subgroups of $W$. 
Since we are in a $p$-group, $p = 2$, we can use Burnside’s Basis Theorem to determine the number of generators of each subgroup of $Q \times Q$. For a group $G$, we define $\Phi(G) = \bigcap \{ M \mid M \text{ is a maximal subgroup of } G \}$. In our case, namely a 2-group, we know that $\Phi(G) = \langle \{ x^2 \mid x \in G \} \rangle$ ([7] Satz 3.14 P.272). A version of Burnside’s Basis Theorem is given below.

**Theorem 0.1.** ([7])

Let $G$ be a $p$-group with $\frac{|G|}{|\Phi(G)|} = p^d$. Then $G$ is generated by exactly $d$ elements, and no set with less than $d$ elements generates $G$. Every element of $G \setminus \Phi(G)$ belongs to some minimal generating set.

**Consequence 1:** $\frac{G}{\Phi(G)}$ is an elementary abelian $p$-group ([7] Satz 3.14 P.272). Hence, it is isomorphic to a vector space over $GF(p)$.

**Consequence 2:** From linear algebra, we then know that the number of maximal subgroups of $G$ is $\frac{p^d - 1}{p - 1}$.

These consequences will be heavily used throughout the construction of the lattice of $Q \times Q$ in Chapter 2.

### 0.1 Automorphisms of $Q$

In Section 2.3, the reader should know that $Aut(Q) \cong S_4$, where $S_4$ is the symmetric group on 4 numbers. Because these 24 automorphisms will be essential to knowing what group we are considering, we will list them in the following section.

In this section, we list the 24 automorphisms of $Q$. They will be denoted by $\tau_n$, where $1 \leq n \leq 24$, and $x$ and $y$ are the generators for $Q$. The list is given below.

The inner automorphisms are:

- $\tau_1 : x \mapsto x$, $\tau_2 : x \mapsto x$, $\tau_3 : x \mapsto x^{-1}$, $\tau_4 : x \mapsto x^{-1}$
- $y \mapsto y$, $y \mapsto y^{-1}$, $y \mapsto y$, $y \mapsto y^{-1}$

Note that the $Out(Q) \cong S_3$. So, we should have 6 cosets of 4. Hence the outer automorphisms are as follows:
\[\begin{align*}
\tau_5 &: \ x \mapsto x, \quad \tau_6 &: \ x \mapsto x, \quad \tau_7 &: \ x \mapsto x^{-1}, \quad \tau_8 &: \ x \mapsto x^{-1} \\
& \quad \ y \mapsto xy \quad \ y \mapsto xy^{-1} \quad \ y \mapsto x^{-1}y \quad \ y \mapsto x^{-1}y^{-1} \\
\tau_9 &: \ x \mapsto y, \quad \tau_{10} &: \ x \mapsto y^{-1}, \quad \tau_{11} &: \ x \mapsto y, \quad \tau_{12} &: \ x \mapsto y^{-1} \\
& \quad \ y \mapsto x \quad \ y \mapsto x \quad \ y \mapsto x^{-1} \quad \ y \mapsto x^{-1} \\
\tau_{13} &: \ x \mapsto y, \quad \tau_{14} &: \ x \mapsto y^{-1}, \quad \tau_{15} &: \ x \mapsto y, \quad \tau_{16} &: \ x \mapsto y^{-1} \\
& \quad \ y \mapsto xy \quad \ y \mapsto xy^{-1} \quad \ y \mapsto x^{-1}y \quad \ y \mapsto xy \\
\tau_{17} &: \ x \mapsto xy, \quad \tau_{18} &: \ x \mapsto xy^{-1}, \quad \tau_{19} &: \ x \mapsto x^{-1}y, \quad \tau_{20} &: \ x \mapsto xy \\
& \quad \ y \mapsto y \quad \ y \mapsto y^{-1} \quad \ y \mapsto y \quad \ y \mapsto y^{-1} \\
\tau_{21} &: \ x \mapsto xy, \quad \tau_{22} &: \ x \mapsto xy^{-1}, \quad \tau_{23} &: \ x \mapsto x^{-1}y, \quad \tau_{24} &: \ x \mapsto xy \\
& \quad \ y \mapsto x \quad \ y \mapsto x \quad \ y \mapsto x^{-1} \quad \ y \mapsto x^{-1}
\end{align*}\]

0.2 Automorphisms of \(V_4\)

In Section 2.4, the reader should know that \(Aut(V_4) \cong S_3\), where \(V_4\) is the Klein 4-group and \(S_3\) is the symmetric group on 3 numbers. Because these 6 automorphisms will be essential to knowing what group we are considering, we will list them in the following Section.

In this section, a list of the 6 automorphisms of \(V_4\), denoted by \(\beta_p\), where \(1 \leq p \leq 6\), is given below. If one would like, think of \(S_3\) as \(GL(2, 2)\). The automorphisms of \(GL(2, 2)\) are:

\[\begin{align*}
\beta_1 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \beta_2 &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} & \beta_3 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
\beta_4 &= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} & \beta_5 &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \beta_6 &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}
\end{align*}\]
For $V_4 = \langle x, y \rangle$, when writing automorphisms on the right, we then have:

- $\beta_1 : x \mapsto x$, $y \mapsto y$
- $\beta_2 : x \mapsto x$, $y \mapsto xy$
- $\beta_3 : x \mapsto y$, $y \mapsto x$
- $\beta_4 : x \mapsto y$, $y \mapsto xy$
- $\beta_5 : x \mapsto xy$, $y \mapsto y$
- $\beta_6 : x \mapsto xy$, $y \mapsto y$

Section 2.6 concludes the chapter with the subgroup lattice of $Q \times Q$. This subgroup lattice uses the details given in Sections 2.1 – 2.5.

In Section 3.1, we calculate the number of subgroups and their orders in $Q \times D_8$. We do not include the details of how we constructed the subgroup lattice of $Q \times D_8$ for the sake of the reader and because the details would be similar, in nature, to the details in construction of the subgroup lattice of $Q \times Q$. The subgroup lattice of $Q \times D_8$ will be given in Section 3.3.

So, why construct the subgroup lattices of $Q \times Q$ and $Q \times D_8$, two groups of order 64? Our reason for constructing these two subgroup lattices is that there are exactly two extraspecial groups of order 32. They can be recognized as the quotient of the group by the diagonal of the centers; that is, $Q \gamma Q$ and $Q \gamma D_8$. In Section 3.2, we discuss extraspecial groups and give a well known classification of them. This classification will reiterate why we chose these two subgroup lattices. A $p$-group $P$ is extraspecial $[2]$ if $Z(P) = P' = \Phi(P)$.

Chapter 3 ends with the subgroup lattices of $Q \times Q$, $Q \times D_8$, and the two extraspecial groups of order 32 along with information about them. If one is interested in further information about groups of order $2^n$, where $n \leq 6$, then [8] is a great reference. Opening that double oversized book was very motivating and inspirational for many reasons. The authors commitment to and passion for understanding groups of order $2^n$ was revealed by the adversity they endured to complete the work they had started before World War II. Mostly, this book allowed us to see how mathematicians studied groups and subgroup lattices that we are currently interested in. Many of the techniques used to construct the tables and lattices which appear in that book are attributed to Philip Hall.
Chapter 1

Characterization of Containment of Subgroups of a Direct Product $A \times B$

Given finite groups $G$ and $H$ and information on the subgroups of $G$ and $H$, what can one say about the subgroups of $G \times H$? In this chapter, we give a characterization of containment of subgroups in a direct product.

In Section 1.1, we state a well known result by Edouard Goursat which gives the subgroup structure of a direct product. In Section 1.2, we present results that will give rise to the mappings and notation introduced in Section 1.3. In Section 1.3, we give the main results, which include a characterization of containment of subgroups. The theorems presented in Section 1.3 are essential in the construction of the subgroup lattices of $Q \times Q$ and $Q \times D_8$, done in Chapters 2 and 3, where $Q$ is the quaternions and $D_8$ is the dihedral group of order 8.

1.1 Goursat’s Theorem

Much is known about subgroups of a direct product and a great deal of the properties that we know about such subgroups depend heavily on applications of Goursat’s theorem. In 1889 [5], Edouard Goursat proved a theorem which describes the subgroup structure of a direct product in terms of the sections of the factor groups.

Theorem 1.1. (Goursat) Let $A$ and $B$ be groups. Then there exists a bijection between the set $S$ of all subgroups of $A \times B$ and the set $T$ of all triples \( \left( \frac{I}{J}, \frac{L}{K}, \sigma \right) \), where $\frac{I}{J}$ is a section of $A$, $\frac{L}{K}$ is a section of $B$ and $\sigma : \frac{I}{J} \rightarrow \frac{L}{K}$ is an isomorphism.

To prove this theorem, one can make the following observations. The projections $\pi_A : \frac{I}{J}$.
$A \times B \to A$ given by $\pi_A(a, b) = a$ and $\pi_B : A \times B \to B$ given by $\pi_B(a, b) = b$, for $a \in A$ and $b \in B$, are homomorphisms.

Let $U \leq A \times B$. Since $A \times B$ is a direct product and each factor is normal in $A \times B$ we can say that $U \cap A \triangleleft U$ and $U \cap B \triangleleft U$. Furthermore since $\pi_A$ and $\pi_B$ are projections we can in fact say that given $U \leq A \times B$, there is a unique triple which corresponds to it. Namely, since $U \leq A \times B$, $U \cap A \triangleleft \pi_A(U)$ and $U \cap B \triangleleft \pi_B(U)$. Let $(a, b) \in U$. Then $\sigma : \frac{\pi_A(U)}{U \cap A} \to \frac{\pi_B(U)}{U \cap B}$ given by $(a(U \cap A))^\sigma = b(U \cap B)$ is an isomorphism. Then one can show that $U = \{(a, b) | a \in \pi_A(U), b \in \pi_B(U), (a(U \cap A))^\sigma = b(U \cap B)\}$. Conversely, given a triple, one obtains $U$ as defined above.

When $U \leq A \times B$, we use the following notation: $I = \pi_A(U)$, $J = U \cap A$, $L = \pi_B(U)$ and $K = U \cap B$.

In proving Goursat’s Theorem, one sees that the subgroups of $A \times B$ are really established by their projections onto $A$ and $B$ and their intersections with $A$ and $B$. So, it is then reasonable to believe that these projections and intersections will come up often when we discuss containment of subgroups in a direct product.

One can use this theorem in counting the number of subgroups of a direct product; see [10] on how to do this. Provided that one is able to determine all the sections of $A$ and $B$ and the isomorphisms between the isomorphic sections, one can count the number of subgroups by counting the number of triples.

### 1.2 Conditions for Projections, Intersections, and Isomorphisms Between Sections of Direct Products

First, we will provide isomorphism theorems that will motivate the isomorphism introduced in Lemma 1.6. Then we will continue the use of the notation introduced in Section 1.1 to provide conditions for containment of subgroups of a direct product.

#### 1.2.1 Background on Isomorphism Theorems

In this section, we provide isomorphism theorems that will motivate the notation introduced in Lemma 1.6. If $\varphi : G \to H$ is a homomorphism, we will denote the inverse image of a
subgroup $N$ as $N^{\varphi^{-1}}$.

**Theorem 1.2.** [12]

If $\varphi : G \to H$ is a homomorphism, $G^\varphi = H$, $K \vartriangleleft H$ and $N = K^{\varphi^{-1}}$, then $\bar{\varphi} : \frac{G}{N} \to \frac{H}{K}$ given by $gN = g^\varphi K$, where $g \in G$, is an isomorphism.

We now recall the following result from Dummit and Foote.

**Theorem 1.3.** [3]

Let $G$ be a group and let $N$ and $M$ be normal subgroups of $G$ with $M \leq N$. Then $\frac{N}{M} \triangleleft \frac{G}{M}$ and $\frac{G}{M} \cong \frac{N}{M}$.

In conjunction with Goursat’s theorem, we use these results to establish notation for the rest of the dissertation.

**FACT:** If $\varphi : G \to H$ is an isomorphism and $K \vartriangleleft H$ with $N = K^{\varphi^{-1}}$, then $N^\varphi = K$ and there is an isomorphism $\bar{\varphi} : \frac{G}{N} \to \frac{H}{N^\varphi}$, as specified in Theorem 1.2. In addition, using Theorem 1.3, if $M \vartriangleleft G$, $M \leq N$, then there is another isomorphism $\hat{\varphi} : \frac{G}{M} \cong \frac{N}{M}$ given by $\left( gM \left( \frac{N}{M} \right) \right)^\varphi = g^\varphi M^\varphi \left( \frac{N}{M} \right)^\varphi$.

### 1.2.2 Lemmas

The goal of this section is to present results that will be used to prove the main results of this chapter. Using the notation introduced in Goursat’s theorem, which describes the projections, intersections, isomorphisms and subgroups of $A \times B$, we will provide background information that will be an integral part of the proof of the main results to follow in section 1.3.

Recall the following notation introduced in Goursat’s theorem. For $U_1$, $U_2 \leq A \times B$, $I_n = \pi_A(U_n)$, $J_n = U_n \cap A$, $L_n = \pi_B(U_n)$ and $K_n = U_n \cap B$, where

$$U_n = \{(a, b) | a \in I_n, b \in L_n, (aJ_n)^{\sigma_n} = bK_n\}$$

for $n = 1, 2$. To further understand the subgroup structure of a direct product we will examine the isomorphisms $\sigma_n : \frac{I_n}{J_n} \to \frac{L_n}{K_n}$.
Lemma 1.4. For \( n \in \{1, 2\} \) let \( U_n \leq A \times B \), \( I_n = \pi_A(U_n) \), \( J_n = U_n \cap A \), \( L_n = \pi_B(U_n) \), \( K_n = U_n \cap B \) and \( \sigma_1 : \frac{I_1}{J_1} \rightarrow \frac{L_1}{K_1} \) be the isomorphism from the triple associated to \( U_1 \). Suppose further that the following conditions hold:

(i) \( I_2 \leq I_1 \)

(ii) \( L_2 \leq L_1 \) and

(iii) \( \left( \frac{I_2 J_1}{J_1} \right)^{\sigma_1} = \frac{L_2 K_1}{K_1} \).

Define \( \tilde{\sigma}_1 : \frac{I_2 J_1}{J_1} \rightarrow \frac{L_2 K_1}{K_1} \) to be the restriction of \( \sigma_1 \). Then \( \tilde{\sigma}_1 \) is an isomorphism.

Lemma 1.5. For \( n = 1, 2 \) suppose \( U_n \leq A \times B \) with \( U_2 \leq U_1 \). Suppose further that \( I_n = \pi_A(U_n) \), \( J_n = U_n \cap A \), \( L_n = \pi_B(U_n) \) and \( K_n = U_n \cap B \). Then

(i) \( J_2 \leq I_2 \cap J_1 \)

(ii) \( K_2 \leq L_2 \cap K_1 \).

The proof of this lemma is routine. Parts (i) and (ii) are easy consequences of the following:

\( U_2 \leq U_1 \) implies that \( I_2 \leq I_1 \), \( L_2 \leq L_1 \), \( J_2 \leq J_1 \) and \( K_2 \leq K_1 \).

Observation: To understand the following lemma it suffices to make a couple observations. First, observe that for \( n \in \{1, 2\} \), \( I_n \triangleleft J_n \), \( I_2 \leq I_1 \) and so \( I_2 \cap J_1 \triangleleft I_2 \) with \( J_2 \leq I_2 \cap J_1 \). Then by Theorem 1.3, \( \frac{I_2 J_1}{J_2} \triangleleft \frac{I_2}{J_2} \) and the mapping \( \eta_A : \left( \frac{I_2}{J_2} \right) / \left( \frac{I_2 J_1}{J_2} \right) \rightarrow \frac{I_2}{J_2} \) is an isomorphism. Second, making a similar observation in \( B \), we will have \( \eta_B : \left( \frac{L_2 K_1}{K_2} \right) / \left( \frac{L_2}{K_2} \right) \rightarrow \frac{L_2}{K_2} \) is an isomorphism.

Lemma 1.6. For \( n = 1, 2 \) let \( U_n \leq A \times B \) with \( U_2 \leq U_1 \). Suppose further that \( I_n = \pi_A(U_n) \), \( J_n = U_n \cap A \), \( L_n = \pi_B(U_n) \), \( K_n = U_n \cap B \) and \( \sigma_2 : \frac{I_2}{J_2} \rightarrow \frac{L_2}{K_2} \) is an isomorphism such that

\( \left( \frac{I_2 J_1}{J_2} \right)^{\sigma_2} = \frac{L_2 K_1}{K_2} \). Then, from the FACT following Theorems 1.2 and 1.3,

(i) There is an isomorphism \( \tilde{\sigma}_2 : \left( \frac{I_2}{J_2} \right) / \left( \frac{I_2 J_1}{J_2} \right) \rightarrow \left( \frac{L_2}{K_2} \right) / \left( \frac{L_2 K_1}{K_2} \right) \) as defined in those consequences.

(ii) We define \( \tilde{\sigma}_2 : \frac{I_2}{I_2 J_1} \rightarrow \frac{L_2}{L_2 K_1} \) by \( \tilde{\sigma}_2 = \eta_A^{-1} \tilde{\sigma}_2 \eta_B \), where composition is read left to right, and it is an isomorphism. (This is a slight deviation from notation in FACT.)

Note that for \((a, b) \in U_2\), \((a(I_2 J_1))^{\tilde{\sigma}_2} = b(L_2 K_1)\).
1.3 Main Results: Containment of Subgroups of a Direct Product

The goal of this section is to give a characterization of containment of subgroups in a product of groups. This is accomplished in Theorem 1.7. In Theorem 1.8, we then state when two subgroups of a direct product are isomorphic under an automorphism of that direct product.

Recall the following. for \( n = 1, 2 \), let \( U_n \leq A \times B \). Suppose further that \( I_2 \leq I_1, J_2 \leq J_1, L_2 \leq L_1 \) and \( K_2 \leq K_1 \). Notice that \( J_1 \triangleleft I_1 \) and \( K_1 \triangleleft L_1 \). Then observe that the mapping \( \theta_1 : \frac{I_2J_1}{J_1} \to \frac{I_2}{I_2 \cap J_1} \) given by \( (aJ_1)^{\theta_1} = a(I_2 \cap J_1) \) is an isomorphism. Similarly, observe that \( \theta_2 : \frac{L_2K_1}{K_1} \to \frac{L_2}{L_2 \cap K_1} \) given by \( (bK_1)^{\theta_2} = b(L_2 \cap K_1) \) is an isomorphism.

**Theorem 1.7.** Suppose \( U_1, U_2 \leq A \times B \). Then \( U_2 \leq U_1 \iff \)

\( (i) \) \( I_2 \leq I_1, J_2 \leq J_1, L_2 \leq L_1 \) and \( K_2 \leq K_1 \)

\( (ii) \) \( \left( \frac{I_2J_1}{J_1} \right)^{\sigma_1} = \frac{L_2K_1}{K_1} \)

\( (iii) \) \( \left( \frac{I_2 \cap J_1}{J_2} \right)^{\sigma_2} = \frac{L_2 \cap K_1}{K_2} \) and

\( (iv) \) \( \tilde{\sigma}_2 \circ \theta_1 = \theta_2 \circ \tilde{\sigma}_1 \), where \( \theta_n \) and \( \tilde{\sigma}_n \) are defined as before and \( n = 1, 2 \).

An equivalent statement for \( (iv) \) is that the following diagram is a commutative diagram.

\[
\begin{array}{ccc}
I_2J_1 & \xrightarrow{\sigma_1} & L_2K_1 \\
\downarrow{\theta_1} & & \downarrow{\theta_2} \\
I_2 \cap J_1 & \xrightarrow{\tilde{\sigma}_2} & L_2 \cap K_1
\end{array}
\]

**Proof:** (\( \implies \)) Suppose \( U_2 \leq U_1 \). It is obvious that \( I_2 \leq I_1, J_2 \leq J_1, L_2 \leq L_1, K_2 \leq K_1, \)

\( \left( \frac{I_2J_1}{J_1} \right)^{\sigma_1} = \frac{L_2K_1}{K_1} \), and \( \left( \frac{I_2 \cap J_1}{J_2} \right)^{\sigma_2} = \frac{L_2 \cap K_1}{K_2} \). Hence, it suffices to show that the diagram commutes. More specifically, that \( \theta_2 \circ \tilde{\sigma}_1 = \tilde{\sigma}_2 \circ \theta_1 \). Let \( c \in I_2 = \pi_A(U_2) \).

Then there exists a \( d \in L_2 = \pi_B(U_2) \) such that \( (c,d) \in U_2 \). Since \( U_2 \leq U_1 \), we know \( (cJ_1)^{\sigma_1} = dK_1 \). By Lemma 1.4, \( \tilde{\sigma}_1 \) is a restriction of \( \sigma_1 \). Hence, we can consider \( (cJ_1)^{\tilde{\sigma}_1} \).

Then \( ((cJ_1)^{\tilde{\sigma}_1})^\theta_2 = (dK_1)^{\theta_2} = d(L_2 \cap K_1) \). On the other hand, using Lemma 1.6, we get \( ((cJ_1)^{\theta_1})^{\tilde{\sigma}_2} = (c(I_2 \cap J_1))^{\tilde{\sigma}_2} = d(L_2 \cap K_1) \). Therefore, \( \theta_2 \circ \tilde{\sigma}_1 = \tilde{\sigma}_2 \circ \theta_1 \), and the diagram commutes.

(\( \iff \)) Conversely, suppose the containments hold and the diagram commutes. Our aim is
to show $U_2 \leq U_1$. Let $(c,d) \in U_2$. Then $(cJ_2)^{\sigma_2} = dK_2$, where $c \in I_2$ and $d \in L_2$. Consider $cJ_1$. We can consider this coset because the containments hold. Because of Lemma 1.6 we can consider $(cJ_1)^{\sigma_2}$. Then $(cJ_1)^{\tilde{\sigma}_1} = (((cJ_1)^{\theta_1})^{\tilde{\sigma}_2})^{\theta_2^{-1}} = ((cI_2 \cap J_1))^{\tilde{\sigma}_2})^{\theta_2^{-1}}$. Using Lemma 1.6, we then see that $((cI_2 \cap J_1))^{\tilde{\sigma}_2})^{\theta_2^{-1}} = (d(L_2 \cap K_1))^{\theta_2^{-1}} = dK_1$. So, $(cJ_1)^{\tilde{\sigma}_1} = dK_1$. Therefore, by Lemma 1.4 we know $(cJ_1)^{\sigma_1} = dK_1$ and $U_2 \leq U_1$.

**Theorem 1.8.** Suppose $U_1, U_2 \leq A \times B$ with $J_n \triangleleft I_n$, $K_n \triangleleft L_n$ and $\sigma_n : I_n/J_n \to L_n/K_n$ such that $U_n = \{(a,b) | a \in I_n, b \in L_n, (aJ_n)^{\sigma_n} = bK_n\}$, $n = 1, 2$. Suppose further that $\alpha \in \text{Aut}(A)$, $\gamma \in \text{Aut}(B)$ such that

$$
\alpha : I_1 \to I_2 \text{ and } J_1 \to J_2
\gamma : L_1 \to L_2 \text{ and } K_1 \to K_2
\overline{\alpha} : I_1/J_1 \to I_2/J_2
\overline{\gamma} : L_1/K_1 \to L_2/K_2
$$

such that $\sigma_2\overline{\alpha} = \overline{\gamma}\sigma_1$. Then $(\alpha, \gamma) \in \text{Aut}(A \times B)$ and $U_1^{(\alpha, \gamma)} = U_2$.

The proof of this theorem is routine to check.

**Remark:** Later we will point out that conditions $(i)-(iv)$ of Theorem 1.7 are independent. This will be done at the end of Section 2.3.
Chapter 2

Construction of the Subgroup Lattice for $Q \times Q$

In this chapter, we apply the theorems from Section 1.3 to construct the subgroup lattice of $Q \times Q$. Throughout each section, there will be a detailed explanation of the group structures of specific orders included with the lattice construction. If one is interested in the lattice only, see the tables with groups and maximal subgroups in each section. As one will see, every group in the construction is determined by certain projections and intersections in each factor of $Q \times Q$. To determine the maximal subgroups of these groups, we will verify the hypothesis of Theorem 1.7. If the sections of each factor of $Q \times Q$ are trivial then the showing that the diagram commutes is trivial, and we only need to verify the other hypotheses of Theorem 1.7. If the sections of each factor are nontrivial then we must verify all of the hypotheses of Theorem 1.7.

In Section 2.1, we will introduce $Q \times Q$ and describe how we used Goursat’s theorem to calculate the number of subgroups. Section 2.2 will introduce the groups of order 2 and the groups of order 4. Section 2.3 will introduce the groups of order 8 and determine the maximal subgroups of each group. Section 2.4 will introduce the groups of order 16 and determine the maximal subgroups of each group. Section 2.5 will introduce the groups of order 32 and determine the maximal subgroups of each group. Section 2.6 will show the subgroup lattice of $Q \times Q$. Let $H$ be a subgroup in $Q \times Q$. In the subgroup lattice the subgroups that were called $\overline{H}$ will be written as $^*H$. 
2.1 Using Goursat to Calculate the Number of Subgroups of \( Q \times Q \)

In the following example we will use the quaternion group of order 8, written \( Q \), to construct the subgroup lattice of \( Q \times Q \). Although \( Q \) is a well known group, it suffices to introduce our notation for it and its subgroups because it will be used heavily and consistently throughout the remainder of the chapter.

The presentation of the quaternions is \( Q = \langle x, y \mid x^4 = y^4 = 1, x^2 = y^2 \rangle \). There are 3 cyclic groups of order 4 contained in \( Q \). Namely, \( F_1 = \langle x \rangle \), \( F_2 = \langle y \rangle \) and \( F_3 = \langle xy \rangle \). There is one cyclic group of order 2 contained in \( Q \), and we will call it \( Z = \langle z \rangle \), where \( z = x^2 \). The subgroup lattice of \( Q \) can be found below.

\[
\begin{array}{c}
Q \\
| \\
F_1 \\
| \\
F_3 \\
| \\
F_2 \\
| \\
Z \\
| \\
1
\end{array}
\]

**Example.** Using the notation for \( Q \) introduced above, let \( E = \{ Q, F_i, Z, 1 \mid 1 \leq i \leq 3 \} \). Our aim is to construct the subgroup lattice for \( G = Q \times Q \). Using Goursat’s Theorem, it first suffices to find the number of subgroups in \( G \) by examining the sections \( I/J \) and \( L/K \), where \( I, J, K, L \in E \). In order to calculate the orders of the subgroups we will use the following:

If \( U \leq A \times B \), then \( |U| = \frac{|I||L|}{|I/J|} \).

**Case 1.** Let \( I = Q \), and let \( J \in E \). Then
<table>
<thead>
<tr>
<th>$I, J$</th>
<th>$I/J$</th>
<th>$\text{Aut}(I/J)$</th>
<th>$L, K$</th>
<th>No. of subgroups</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q, Q$</td>
<td>1</td>
<td>1</td>
<td>$Q, Q$</td>
<td>1 of order 64</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$F_j, F_j$</td>
<td>3 of order 32</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$Z, Z$</td>
<td>1 of order 16</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1, 1</td>
<td>1 of order 8</td>
</tr>
<tr>
<td>$Q, F_i$</td>
<td>$C_2$</td>
<td>1</td>
<td>$Q, F_j$</td>
<td>9 of order 32</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$F_j, Z$</td>
<td>9 of order 16</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$Z, 1$</td>
<td>3 of order 8</td>
</tr>
<tr>
<td>$Q, Z$</td>
<td>$V_4$</td>
<td>$S_3$</td>
<td>$Q, Z$</td>
<td>6 of order 16</td>
</tr>
<tr>
<td>$Q, 1$</td>
<td>$Q$</td>
<td>$S_4$</td>
<td>$Q, 1$</td>
<td>24 of order 8</td>
</tr>
</tbody>
</table>

**Case 2.** Let $I = F_i$, and let $J \in E \setminus \{Q\}$. Then

<table>
<thead>
<tr>
<th>$I, J$</th>
<th>$I/J$</th>
<th>$\text{Aut}(I/J)$</th>
<th>$L, K$</th>
<th>No. of subgroups</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_i, F_i$</td>
<td>1</td>
<td>1</td>
<td>$Q, Q$</td>
<td>3 of order 32</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$F_j, F_j$</td>
<td>9 of order 16</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$Z, Z$</td>
<td>3 of order 8</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1, 1</td>
<td>3 of order 4</td>
</tr>
<tr>
<td>$F_i, Z$</td>
<td>$C_2$</td>
<td>1</td>
<td>$Q, F_j$</td>
<td>9 of order 16</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$F_j, Z$</td>
<td>9 of order 8</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$Z, 1$</td>
<td>3 of order 4</td>
</tr>
<tr>
<td>$F_i, 1$</td>
<td>$C_4$</td>
<td>$C_2$</td>
<td>$F_j, 1$</td>
<td>18 of order 4</td>
</tr>
</tbody>
</table>

**Case 3.** Let $I = Z$, and let $J \in E \setminus \{Q, F_i\}$. Then

<table>
<thead>
<tr>
<th>$I, J$</th>
<th>$I/J$</th>
<th>$\text{Aut}(I/J)$</th>
<th>$L, K$</th>
<th>No. of subgroups</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z, Z$</td>
<td>1</td>
<td>1</td>
<td>$Q, Q$</td>
<td>1 of order 16</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$F_j, F_j$</td>
<td>3 of order 8</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$Z, Z$</td>
<td>1 of order 4</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1, 1</td>
<td>1 of order 2</td>
</tr>
<tr>
<td>$Z, 1$</td>
<td>$C_2$</td>
<td>1</td>
<td>$Q, F_j$</td>
<td>3 of order 8</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$F_j, Z$</td>
<td>3 of order 4</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$Z, 1$</td>
<td>1 of order 2</td>
</tr>
</tbody>
</table>

**Case 4.** Let $I = 1 = J$. Then
2.2 The Subgroups of Orders 2 and 4

In order to construct this subgroup lattice, it suffices to introduce a bit of notation and to determine the maximal subgroups on each level. Clearly, the groups of order 2 are all copies of $C_2$, and the only subgroup it contains is the trivial subgroup. For the sake of construction and reference, the groups are listed in the table below.

**Subgroups of Order 2**

<table>
<thead>
<tr>
<th>Notation</th>
<th>$I$, $J$</th>
<th>$I/J$</th>
<th>$Aut(I/J)$</th>
<th>$L$, $K$</th>
<th>No. of subgroups</th>
<th>group structure</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z_1$</td>
<td>$Z$, $Z$</td>
<td>1</td>
<td>1</td>
<td>1, 1</td>
<td>1</td>
<td>$C_2$</td>
</tr>
<tr>
<td>$Z_2$</td>
<td>$Z$, 1</td>
<td>$C_2$</td>
<td>1</td>
<td>$Z$, 1</td>
<td>1</td>
<td>$C_2$</td>
</tr>
<tr>
<td>$Z_3$</td>
<td>1, 1</td>
<td>1</td>
<td>1</td>
<td>$Z$, $Z$</td>
<td>1</td>
<td>$C_2$</td>
</tr>
</tbody>
</table>

**Subgroups of Order 4**

The groups of order 4 are either copies of $C_4$, the cyclic group of order 4, or $V_4$ the Klein-4 group. The groups of order 4 are given in the table below, and for these groups $1 \leq i, j \leq 3$.

<table>
<thead>
<tr>
<th>Notation</th>
<th>$I$, $J$</th>
<th>$I/J$</th>
<th>$Aut(I/J)$</th>
<th>$L$, $K$</th>
<th>No. of subgroups</th>
<th>group structure</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\overline{F}_i$</td>
<td>$F_i$, $F_i$</td>
<td>1</td>
<td>1</td>
<td>1, 1</td>
<td>3</td>
<td>$C_4$</td>
</tr>
<tr>
<td>$\overline{F}_j$</td>
<td>1, 1</td>
<td>1</td>
<td>1</td>
<td>$F_j$, $F_j$</td>
<td>3</td>
<td>$C_4$</td>
</tr>
<tr>
<td>$\overline{X}_i$</td>
<td>$F_i$, $Z$</td>
<td>$C_2$</td>
<td>1</td>
<td>$Z$, 1</td>
<td>3</td>
<td>$C_4$</td>
</tr>
<tr>
<td>$\overline{X}_j$</td>
<td>$Z$, 1</td>
<td>$C_2$</td>
<td>1</td>
<td>$F_j$, $Z$</td>
<td>3</td>
<td>$C_4$</td>
</tr>
<tr>
<td>$\overline{F}<em>{ij}$, $\overline{F}</em>{ij}$</td>
<td>$F_i$, 1</td>
<td>$C_4$</td>
<td>$C_2$</td>
<td>$F_j$, 1</td>
<td>18</td>
<td>$C_4$</td>
</tr>
<tr>
<td>$V$</td>
<td>$Z$, $Z$</td>
<td>1</td>
<td>1</td>
<td>$Z$, $Z$</td>
<td>1</td>
<td>$V_4$</td>
</tr>
</tbody>
</table>
Observe that all of the groups of order 4 are copies of \( C_4 \), with the exception of \( V \). To verify these group structures, one can examine the generators of the group in question. Then determining its maximal subgroup(s) is a routine check, consisting of verifying the hypotheses of Theorem 1.7.

For the groups \( \overrightarrow{F}_{ij} \) and \( \overrightarrow{F}_{ij}^\prime \), \( Aut(I/J) = C_2 \). \( \overrightarrow{F}_{ij} \) is the group that corresponds to the identity automorphism, and \( \overrightarrow{F}_{ij}^\prime \) is the group that corresponds to the automorphism which sends \( x \) to \( x^{-1} \). As an example of how one would verify the group structure of one of these groups of order 4 and determine its maximal subgroup(s), read below. Otherwise, one can examine the table at the end of this section to see a list of the maximal subgroups of the groups of order 4.

Consider \( \overrightarrow{F}_{11} \). In this case, \( \sigma_1 \) is the isomorphism that maps \( F_1 \to F_1 \). So, \( \overrightarrow{F}_{11} = \{(x,x)|x \in F_1\} \) is a cyclic group of order 4, written \( C_4 \). To determine its maximal subgroup we could verify the hypotheses of Theorem 1.7. However, since this group is cyclic of order 4, notice that its unique subgroup of order 2 is generated by the square of \( (x,x) \), and this group is \( Z_2 \) given in the section discussing groups of order 2.

Let us now explain why \( V \cong V_4 \), where \( V_4 \cong C_2 \times C_2 \). Notice that \( V = \{(a,b)|a \in Z, b \in Z\} \cong C_2 \times C_2 \). This group has 3 maximal subgroups of order 2, and they are \( Z_1 \), \( Z_2 \) and \( Z_3 \). The maximal subgroups of all the groups of order 4 are listed in the table below.

<table>
<thead>
<tr>
<th>Groups</th>
<th>Maximal Subgroups</th>
<th>Groups</th>
<th>Maximal Subgroups</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \overrightarrow{F}_i )</td>
<td>( Z_1 )</td>
<td>( \overrightarrow{F}_j )</td>
<td>( Z_3 )</td>
</tr>
<tr>
<td>( \overrightarrow{X}_i )</td>
<td>( Z_1 )</td>
<td>( \overrightarrow{X}_i )</td>
<td>( Z_3 )</td>
</tr>
<tr>
<td>( \overrightarrow{F}_{ij} )</td>
<td>( Z_2 )</td>
<td>( \overrightarrow{F}_{ij} )</td>
<td>( Z_2 )</td>
</tr>
<tr>
<td>( V )</td>
<td>( Z_1, Z_2, Z_3 )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2.3 The Subgroups of Order 8

The groups of order 8 that appear in this lattice will be copies of \( Q \), \( C_4 \times C_2 \) or \( C_2 \times C_4 \). We are familiar with the subgroup lattice of \( Q \). In order to identify \( C_4 \times C_2 \) in \( Q \times Q \), it suffices to give its subgroup lattice. It is given below.
Although not used within the explanations to follow, knowing the subgroup lattice above was very helpful in determining and checking that the groups with the above group structure indeed had the correct maximal subgroups. The table that follows lists the groups of order 8 and its corresponding group structure. For these groups, $1 \leq i, j \leq 3$ and $1 \leq n \leq 24$.

<table>
<thead>
<tr>
<th>Notation</th>
<th>$I, J$</th>
<th>$I/J$</th>
<th>$Aut(I/J)$</th>
<th>$L, K$</th>
<th>No. of subgroups</th>
<th>group structure</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\overrightarrow{Q}_1$</td>
<td>$Q, Q$</td>
<td>1, 1</td>
<td>1, 1</td>
<td>1</td>
<td>$Q$</td>
<td></td>
</tr>
<tr>
<td>$\overrightarrow{Q}_2$</td>
<td>1, 1</td>
<td>1, 1</td>
<td>$Q, Q$</td>
<td>1</td>
<td>$Q$</td>
<td></td>
</tr>
<tr>
<td>$\overrightarrow{W}_i$</td>
<td>$Q, F_i$</td>
<td>$C_2$</td>
<td>1</td>
<td>$Z, 1$</td>
<td>3</td>
<td>$Q$</td>
</tr>
<tr>
<td>$\overrightarrow{W}_j$</td>
<td>$Z, 1$</td>
<td>$C_2$</td>
<td>1</td>
<td>$Q, F_j$</td>
<td>3</td>
<td>$Q$</td>
</tr>
<tr>
<td>$\overrightarrow{Y}_i$</td>
<td>$F_i, F_i$</td>
<td>1, 1</td>
<td>$Z, Z$</td>
<td>3</td>
<td>$C_4 \times C_2$</td>
<td></td>
</tr>
<tr>
<td>$\overrightarrow{Y}_j$</td>
<td>$Z, Z$</td>
<td>1, 1</td>
<td>$F_j, F_j$</td>
<td>3</td>
<td>$C_2 \times C_4$</td>
<td></td>
</tr>
<tr>
<td>$F_{ij}$</td>
<td>$F_i, Z$</td>
<td>$C_2$</td>
<td>1</td>
<td>$F_j, Z$</td>
<td>9</td>
<td>$C_4 \times C_2$</td>
</tr>
<tr>
<td>$\Delta_n$</td>
<td>$Q, 1$</td>
<td>$Q$</td>
<td>$S_4$</td>
<td>24</td>
<td>$Q$</td>
<td></td>
</tr>
</tbody>
</table>

Now, let us begin to determine the maximal subgroups of the groups of order 8. Using the table above, we will begin by determining the maximal subgroups of the groups that have trivial sections, and then we will determine the maximal subgroups of the groups that have nontrivial sections.

Consider $\overrightarrow{Q}_1 = Q \times 1$. Its maximal subgroups are $\overrightarrow{F}_i = F_i \times 1$, $1 \leq i \leq 3$. Similarly, $\overrightarrow{Q}_2 = 1 \times Q$, and its maximal subgroups are $\overrightarrow{F}_j = 1 \times F_j$, $1 \leq j \leq 3$. A table with the maximal subgroups of $\overrightarrow{Q}_1$ and $\overrightarrow{Q}_2$ is given below.

<table>
<thead>
<tr>
<th>Groups</th>
<th>Maximal Subgroups</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\overrightarrow{Q}_1$</td>
<td>$\overrightarrow{F}_1, \overrightarrow{F}_2, \overrightarrow{F}_3$</td>
</tr>
<tr>
<td>$\overrightarrow{Q}_2$</td>
<td>$\overrightarrow{F}_1, \overrightarrow{F}_2, \overrightarrow{F}_3$</td>
</tr>
</tbody>
</table>

Consider $\overrightarrow{Y}_1$, which is clearly isomorphic to $C_2 \times C_4$. As shown at the beginning of this
section, \(C_2 \times C_4\) has 3 maximal subgroups, and we claim that the maximal subgroups of \(\vec{Y}_1\) are \(\vec{X}_1\), \(\vec{F}_1\) and \(V\). Since these groups have order 4, they are indeed maximal. To determine that these are subgroups, we must verify the hypotheses of Theorem 1.7. Examine the tables below to see that the containment of the projections and intersections of these proposed maximal subgroups are satisfied.

<table>
<thead>
<tr>
<th>(\vec{X}_1)</th>
<th>(\vec{Y}_1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(I_2 = Z)</td>
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</tr>
<tr>
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<td>(J_1 = Z)</td>
</tr>
<tr>
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</tr>
<tr>
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<td>(K_1 = F_1)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(\vec{F}_1)</th>
<th>(\vec{Y}_1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(I_2 = 1)</td>
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</tr>
<tr>
<td>(J_2 = 1)</td>
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</tr>
<tr>
<td>(L_2 = F_1)</td>
<td>(L_1 = F_1)</td>
</tr>
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<td>(K_2 = F_1)</td>
<td>(K_1 = F_1)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(V)</th>
<th>(\vec{Y}_1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(I_2 = Z)</td>
<td>(I_1 = Z)</td>
</tr>
<tr>
<td>(J_2 = Z)</td>
<td>(J_1 = Z)</td>
</tr>
<tr>
<td>(L_2 = Z)</td>
<td>(L_1 = F_1)</td>
</tr>
<tr>
<td>(K_2 = Z)</td>
<td>(K_1 = F_1)</td>
</tr>
</tbody>
</table>

Now, we must show that parts (ii) and (iii) of Theorem 1.7, namely \(\left(\frac{I_2 J_1}{J_2}\right)^{\sigma_1} = \frac{L_2 K_1}{K_2}\) and \(\left(\frac{I_2 \cap J_1}{J_2}\right)^{\sigma_2} = \frac{L_2 \cap K_1}{K_2}\) respectively, are satisfied for each proposed maximal subgroup. In order to do this, we must know \(\sigma_1\) and \(\sigma_2\). In our case, \(\sigma_1 : \frac{Z}{Z} \to \frac{F_1}{F_1}\), which is the identity, and verifying part (ii) of Theorem 1.7 is trivial. Observe that \(\sigma_2\) changes for each subgroup that we are considering. Hence, it suffices to examine each of the proposed maximal subgroups, determine \(\sigma_2\) and verify part (iii) of Theorem 1.7.

For \(\vec{X}_1\), \(\sigma_2 : \frac{Z}{1} \to \frac{F_1}{Z}\) and \((z)^{\sigma_2} = xZ\). Then \(\left(\frac{I_2 \cap J_1}{J_2}\right)^{\sigma_2} = \left(\frac{Z \cap Z}{1}\right)^{\sigma_2} = (Z)^{\sigma_2} = (xZ)\) and \(\frac{L_2 \cap K_1}{K_2} = \frac{F_1 \cap F_1}{Z} = F_1 Z = (xZ)\). Hence, part (iii) is satisfied.

For \(\vec{F}_1\) and \(V\), the associated \(\sigma_2\) is the identity map. Hence, part (iii) is obviously satisfied.
Showing that the diagram commutes is trivial because the sections which define $\overrightarrow{Y_1}$ are trivial. Therefore, the hypotheses of Theorem 1.7 are satisfied, and the maximal subgroups of $\overrightarrow{Y_1}$ are $\overrightarrow{X_1}$, $\overrightarrow{F_1}$ and $V$.

One can show that $\overrightarrow{Y_j}$ are isomorphic, where $1 \leq j \leq 3$, by applying Theorem 1.8. Namely, one can use an automorphism of $Q \times Q$ to show that the subgroups are isomorphic. Let us now show that $\overrightarrow{Y_1} \cong \overrightarrow{Y_2}$. The associated $\tau_1$ for $\overrightarrow{Y_1}$ and $\overrightarrow{Y_2}$ is the identity. So, to show $\overrightarrow{Y_1} \cong \overrightarrow{Y_2}$, we need to recall $\tau_0 \in Aut(Q)$ given by $(x)^{\tau_0} = y$. Then $(1, \tau_0) \in Aut(Q \times Q)$ and $(\overrightarrow{Y_1})^{(1,\tau_0)} = \overrightarrow{Y_2}$. Similarly, to show that $\overrightarrow{Y_1} \cong \overrightarrow{Y_3}$, recall that $\tau_{17} \in Aut(Q)$ given by $(x)^{\tau_{17}} = xy$. Then $(1, \tau_{17}) \in Aut(Q \times Q)$ and $(\overrightarrow{Y_1})^{(1,\tau_{17})} = \overrightarrow{Y_3}$. Hence $\overrightarrow{Y_j}$’s are isomorphic.

To determine the maximal subgroups of $\overrightarrow{Y_2}$ and $\overrightarrow{Y_3}$, we will use the same automorphisms of $Q \times Q$ from above that we used to show $\overrightarrow{Y_j}$ were isomorphic. Then we will examine the maximal subgroups of $\overrightarrow{Y_1}$ under these automorphisms to attain the maximal subgroups of the isomorphic subgroup we are considering. Recall that the maximal subgroups of $\overrightarrow{Y_1}$ are $\overrightarrow{X_1}$, $\overrightarrow{F_1}$ and $V$.

Let us now determine the maximal subgroups of $\overrightarrow{Y_2}$. Consider $\overrightarrow{X_1}$. Then $(\overrightarrow{X_1})^{(1,\tau_0)} = \langle (z, xZ) \rangle = \overrightarrow{X_2}$. Consider $\overrightarrow{F_1}$. Then $(\overrightarrow{F_1})^{(1,\tau_0)} = (1 \times F_1)(1, \tau_0) = 1 \times F_2 = \overrightarrow{F_2}$. Consider $V$. Then $V^{(1,\tau_0)} = \langle (z, z) \rangle = V$. Therefore, the maximal subgroups of $\overrightarrow{Y_2}$ are $\overrightarrow{X_2}$, $\overrightarrow{F_2}$ and $V$. Similarly, using $(1, \tau_{17})$, one can show the maximal subgroups of $\overrightarrow{Y_3}$ are $\overrightarrow{X_3}$, $\overrightarrow{F_3}$ and $V$.

Analogously, we can produce a similar argument for $\overrightarrow{Y_i}$, $1 \leq i \leq 3$, by noticing that this group is defined by reversing the roles of $I, J$ and $L, K$ in $\overrightarrow{Y_j}$. Also, we can make this observation for $\overrightarrow{Y_i}$ by noticing that $\rho : (a, b) \mapsto (b, a)$ is an automorphism of $Q \times Q$. The maximal subgroups of $\overrightarrow{Y_i}$ are $\overrightarrow{X_i}$, $\overrightarrow{F_i}$ and $V$. A table with the maximal subgroups of $\overrightarrow{Y_i}$ and $\overrightarrow{Y_j}$, where $1 \leq i, j \leq 3$, is given below.

<table>
<thead>
<tr>
<th>Groups</th>
<th>Maximal Subgroups</th>
<th>Groups</th>
<th>Maximal Subgroups</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\overrightarrow{Y_1}$</td>
<td>$\overrightarrow{X_1}$, $\overrightarrow{F_1}$, $V$</td>
<td>$\overrightarrow{Y_1}$</td>
<td>$\overrightarrow{X_1}$, $\overrightarrow{F_1}$, $V$</td>
</tr>
<tr>
<td>$\overrightarrow{Y_2}$</td>
<td>$\overrightarrow{X_2}$, $\overrightarrow{F_2}$, $V$</td>
<td>$\overrightarrow{Y_2}$</td>
<td>$\overrightarrow{X_2}$, $\overrightarrow{F_2}$, $V$</td>
</tr>
<tr>
<td>$\overrightarrow{Y_3}$</td>
<td>$\overrightarrow{X_3}$, $\overrightarrow{F_3}$, $V$</td>
<td>$\overrightarrow{Y_3}$</td>
<td>$\overrightarrow{X_3}$, $\overrightarrow{F_3}$, $V$</td>
</tr>
</tbody>
</table>

Consider $\overrightarrow{W_1} = \{(a, b) | a \in Q, b \in Z, (aF_1)^{\sigma_1} = b\}$, where $\sigma_1 : \frac{Q}{F_1} \rightarrow Z$ is given by $(yF_1)^{\sigma_1} = \ldots$.
z. So, \((y, z), (x, 1) \in \hat{W}_1\), \(\hat{W}_1\) is nonabelian, \((y, z)^4 = 1 = (x, 1)^4\) and \((y, z) \neq (x, 1)\). Hence, \(\hat{W}_1 = \langle (y, z), (x, 1) \rangle \cong Q\). We propose that its maximal subgroups are \(\hat{F}_1, \hat{X}_2\) and \(\hat{X}_3\).

To determine that these are subgroups, we must verify the hypotheses of Theorem 1.7. Examine the tables below to see that the containment of the projections and intersections of these proposed maximal subgroups are satisfied.

<table>
<thead>
<tr>
<th>(\hat{F}_1)</th>
<th>(\hat{W}_1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(I_2 = F_1)</td>
<td>(I_1 = Q)</td>
</tr>
<tr>
<td>(J_2 = F_1)</td>
<td>(J_1 = F_1)</td>
</tr>
<tr>
<td>(L_2 = 1)</td>
<td>(L_1 = Z)</td>
</tr>
<tr>
<td>(K_2 = 1)</td>
<td>(K_1 = 1)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(\hat{X}_2)</th>
<th>(\hat{W}_1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(I_2 = F_2)</td>
<td>(I_1 = Q)</td>
</tr>
<tr>
<td>(J_2 = Z)</td>
<td>(J_1 = F_1)</td>
</tr>
<tr>
<td>(L_2 = Z)</td>
<td>(L_1 = Z)</td>
</tr>
<tr>
<td>(K_2 = 1)</td>
<td>(K_1 = 1)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(\hat{X}_3)</th>
<th>(\hat{W}_1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(I_2 = F_3)</td>
<td>(I_1 = Q)</td>
</tr>
<tr>
<td>(J_2 = Z)</td>
<td>(J_1 = F_1)</td>
</tr>
<tr>
<td>(L_2 = Z)</td>
<td>(L_1 = Z)</td>
</tr>
<tr>
<td>(K_2 = 1)</td>
<td>(K_1 = 1)</td>
</tr>
</tbody>
</table>

Now, we must verify that parts (ii), (iii) and (iv), of Theorem 1.7, are satisfied for each proposed maximal subgroup. Namely that \(\left(\frac{I_2 J_1}{J_1}\right)^{\sigma_1} = \frac{L_2 K_1}{K_1}\), \(\left(\frac{I_2 \cap J_1}{J_2}\right)^{\sigma_2} = \frac{L_2 \cap K_1}{K_2}\), and that the diagram commutes respectively. To verify parts (ii) and (iii), we must know \(\sigma_1\) and \(\sigma_2\). In our case, \(\sigma_1 : \frac{Q}{F_1} \rightarrow Z\) and \((yF_1)^{\sigma_1} = z\). Observe that \(\sigma_1\) will remain the same for each proposed maximal subgroup, and \(\sigma_2\) changes. To verify part (iv), that is, \(\tilde{\sigma}_2 \circ \theta_1 = \theta_2 \circ \tilde{\sigma}_1\), for each proposed subgroup we will give the associated maps \(\theta_1, \tilde{\sigma}_2, \tilde{\sigma}_1\) and \(\theta_2\). Then we will verify that the diagram commutes. In verifying part (iv) it is enough to check that the diagram commutes using the generators because our maps are linear. Let us now examine each of the proposed maximal subgroups, determine \(\sigma_2\) and verify parts (ii), (iii) and (iv) of Theorem 1.7.
Consider $\overline{F}_1$. Then \((I_2J_1)^\sigma_i = (F_1F_1)^\sigma_i = 1\) and $\frac{L_2K_1}{K_1} = 1$. So, part (ii) is satisfied.

The associated $\sigma_2$ for $\overline{F}_1$ is the identity, and hence part (iii) is obviously satisfied. To show the diagram commutes for $\overline{F}_1$, observe that $\frac{L_2}{L_2 \cap K_1} = 1$. This means that all of the associated maps must be the identity. One can verify this is indeed true. Therefore, the diagram commutes and the hypotheses of Theorem 1.7 are satisfied. Hence, $\overline{F}_1$ is a maximal subgroup.

Consider $\overline{X}_2$. Then \((I_2J_1)^\sigma_i = (F_2F_1)^\sigma_i = (yF_1)^\sigma_i = (z) = Z\) and $\frac{L_2K_1}{K_1} = Z$.

So, part (ii) is satisfied. Observe that $\sigma_2 : \frac{F_2}{Z} \rightarrow Z$ is given by $(yZ)^\sigma_2 = z$. Then
\[
\left(\frac{I_2 \cap J_1}{J_2}\right)^\sigma_2 = \left(\frac{F_2 \cap F_1}{Z}\right)^\sigma_2 = \left(\frac{Z}{Z}\right)^\sigma_2 = 1 \quad \text{and} \quad \frac{L_2 \cap K_1}{K_2} = \frac{Z \cap 1}{1} = 1.
\]
Hence, part (iii) is satisfied.

For $\overline{X}_2$, we have the following associated maps:
\[
\theta_1 : \frac{F_2F_1}{F_1} = \frac{Q}{F_1} \rightarrow \frac{F_2}{F_2 \cap F_1} = \frac{Z}{Z} \quad \text{is given by} \quad (yF_1)^\theta_1 = yZ;
\]
\[
\sigma_2 : \frac{F_2 \cap F_1}{Z} \rightarrow \frac{Z}{Z} \quad \text{is given by} \quad (yZ)^\sigma_2 = z;
\]
\[
\sigma_1 : \frac{F_2F_1}{F_1} = \frac{Q}{F_1} \rightarrow Z \quad \text{is given by} \quad (yF_1)^\sigma_1 = z;
\]
\[
\theta_2 : Z \rightarrow \frac{Z}{Z} \quad \text{is given by} \quad (z)^\theta_2 = z.
\]

Then $\sigma_2(\theta_1(yF_1)) = \sigma_2(yZ) = z$ and $\theta_2(\sigma_1(yF_1)) = \theta_2(z) = z$. Hence the diagram commutes, and $\overline{X}_2$ is a maximal subgroup.

Consider $\overline{X}_3$. Then \((I_2J_1)^\sigma_i = (F_3F_1)^\sigma_i = (yF_1)^\sigma_i = (z) = Z\) and $\frac{L_2K_1}{K_1} = Z$.

So, part (ii) is satisfied. Observe that $\sigma_2 : \frac{F_3}{Z} \rightarrow Z$ is given by $(xyZ)^\sigma_2 = z$. Then
\[
\left(\frac{I_2 \cap J_1}{J_2}\right)^\sigma_2 = \left(\frac{F_3 \cap F_1}{Z}\right)^\sigma_2 = \left(\frac{Z}{Z}\right)^\sigma_2 = 1 \quad \text{and} \quad \frac{L_2 \cap K_1}{K_2} = \frac{Z \cap 1}{1} = 1.
\]
Hence, part (iii) is satisfied.

For $\overline{X}_3$, we have the following associated maps:
\[
\theta_1 : \frac{F_3F_1}{F_1} = \frac{Q}{F_1} \rightarrow \frac{F_3}{F_3 \cap F_1} = \frac{Z}{Z} \quad \text{is given by} \quad (yF_1)^\theta_1 = xyZ;
\]
\[
\sigma_2 : \frac{F_3 \cap F_1}{Z} \rightarrow \frac{Z}{Z} \quad \text{is given by} \quad (xyZ)^\sigma_2 = z;
\]
\[
\sigma_1 : \frac{F_3F_1}{F_1} = \frac{Q}{F_1} \rightarrow Z \quad \text{is given by} \quad (yF_1)^\sigma_1 = z;
\]
\[
\theta_2 : Z \rightarrow \frac{Z}{Z} \quad \text{is given by} \quad (z)^\theta_2 = z.
\]

Then $\sigma_2(\theta_1(yF_1)) = \sigma_2(xyZ) = z$ and $\theta_2(\sigma_1(yF_1)) = \theta_2(z) = z$. Therefore, the diagram commutes, and the hypotheses of Theorem 1.7 are satisfied for all of our proposed maximal
subgroups. Hence the maximal subgroups of $\hat{W}_1$ are $\hat{F}_1$, $\hat{X}_2$ and $\hat{X}_3$.

One can show that $\hat{W}_i$ are isomorphic, where $1 \leq i \leq 3$, by applying Theorem 1.8. Namely, one can use an automorphism of $Q \times Q$ to show that the subgroups are isomorphic. Let us now show that $\hat{W}_1 \cong \hat{W}_2$. For $\hat{W}_1$, $\sigma_1 : \frac{Q}{F_1} \to Z$ is given by $(yF_1)^{\sigma_1} = z$. For $\hat{W}_2$, $\sigma_1 : \frac{Q}{F_2} \to Z$ is given by $(xF_2)^{\sigma_1} = z$. So, for $\hat{W}_1$ to be isomorphic to $\hat{W}_2$, we need two automorphisms of $Q$. The first automorphism we need should send $y$ to $x$ and $x$ to $y$. The second automorphism we need is just the identity. Recall $\tau_9 \in \text{Aut}(Q) \text{ given by } (y)^{\tau_9} = x$ and $(x)^{\tau_9} = y$. Then $(\tau_9, 1) \in \text{Aut}(Q \times Q)$ and $(\hat{W}_1)^{(\tau_9, 1)} = \hat{W}_2$. To show that $\hat{W}_1 \cong \hat{W}_3$, observe that, for $\hat{W}_3$, $\sigma_1 : \frac{Q}{F_3} \to Z$ and is given by $(yF_3)^{\sigma_1} = z$. Then recall that $\tau_{17} \in \text{Aut}(Q)$ given by $(y)^{\tau_{17}} = y$ and $(x)^{\tau_{17}} = xy$. So, $(\tau_{17}, 1) \in \text{Aut}(Q \times Q)$ and $(\hat{W}_1)^{(\tau_{17}, 1)} = \hat{W}_3$. Hence, $\hat{W}_1$’s are isomorphic.

To determine the maximal subgroups of $\hat{W}_2$ and $\hat{W}_3$, we will use the same automorphisms of $Q \times Q$ from above that we used to show $\hat{W}_1$ were isomorphic. Then we will examine the maximal subgroups of $\hat{W}_1$ under these automorphisms to attain the maximal subgroups of the isomorphic subgroup we are considering. Recall that the maximal subgroups of $\hat{W}_1$ are $\hat{F}_1$, $\hat{X}_2$ and $\hat{X}_3$.

Let us now determine the maximal subgroups of $\hat{W}_2$. Consider $\hat{F}_1$. Then $(\hat{F}_1)^{(\tau_9, 1)} = (F_1 \times 1)^{(\tau_9, 1)} = F_2 \times 1 = \hat{F}_2$. Consider $\hat{X}_2$. Then $(\hat{X}_2)^{(\tau_9, 1)} = \langle (yZ, z) \rangle^{(\tau_9, 1)} = \langle (xZ, z) \rangle = \hat{X}_1$. Consider $\hat{X}_3$. Observe that $(xy)^{\tau_9} = x^{-1}y$, and that $xZ = x^{-1}Z$ in $Q$. Then by our observation, $(\hat{X}_3)^{(\tau_9, 1)} = \langle (xyZ, z) \rangle^{(\tau_9, 1)} = \langle (x^{-1}yZ, z) \rangle = \langle (xyZ, z) \rangle = \hat{X}_3$. Therefore, the maximal subgroups of $\hat{W}_2$ are $\hat{F}_2$, $\hat{X}_1$ and $\hat{X}_3$. Similarly, using $(\tau_{17}, 1)$, one can show the maximal subgroups of $\hat{W}_3$ are $\hat{F}_3$, $\hat{X}_1$ and $\hat{X}_2$.

Analogously, we can produce a similar argument for $\hat{W}_j$, $1 \leq j \leq 3$, by noticing that this group is defined by reversing the roles of $I, J$ and $L, K$ in $\hat{W}_i$. The maximal subgroups of $\hat{W}_j$ are $\hat{F}_j$ and $\hat{X}_k$, where $1 \leq j, k \leq 3$ and $k \neq j$. A table with the maximal subgroups of $\hat{W}_i$ and $\hat{W}_j$ is given below.

<table>
<thead>
<tr>
<th>Groups $\hat{W}_i$</th>
<th>Maximal Subgroups $\hat{F}_1, \hat{X}_2, \hat{X}_3$</th>
<th>Groups $\hat{W}_j$</th>
<th>Maximal Subgroups $\hat{F}_j, \hat{X}_k, \hat{X}_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{W}_1$</td>
<td>$\hat{F}_1, \hat{X}_2, \hat{X}_3$</td>
<td>$\hat{W}_1$</td>
<td>$\hat{F}_1, \hat{X}_2, \hat{X}_3$</td>
</tr>
<tr>
<td>$\hat{W}_2$</td>
<td>$\hat{F}_2, \hat{X}_1, \hat{X}_3$</td>
<td>$\hat{W}_2$</td>
<td>$\hat{F}_2, \hat{X}_1, \hat{X}_3$</td>
</tr>
<tr>
<td>$\hat{W}_3$</td>
<td>$\hat{F}_3, \hat{X}_1, \hat{X}_2$</td>
<td>$\hat{W}_3$</td>
<td>$\hat{F}_3, \hat{X}_1, \hat{X}_2$</td>
</tr>
</tbody>
</table>
Consider $F_{11} = \{(a,b) | a \in F_1, b \in F_1, (aZ)^{\sigma_1} = bZ\}$, where $\sigma_1 : \frac{F_1}{Z} \to \frac{F_1}{Z}$ and $(xZ)^{\sigma_1} = xZ$. So, $(x,x),(z,z) \in F_{11}$, $(x,x)^4 = 1 = (z,z)^2$ and $(x,x) \neq (z,z)$. Hence, $F_{11} = \langle (x,x),(z,z) \rangle \cong C_4 \times C_2$. As discussed at the beginning of this section, this group has 3 maximal subgroups, and we propose that they are $\widetilde{F_{11}}, \overrightarrow{F_{11}}$ and $V$. Recall that $\widetilde{F_{11}}$ is the group which sends $x$ to $x$, and $\overrightarrow{F_{11}}$ is the group which sends $x$ to $x^{-1}$. To determine that these are subgroups, we must verify the hypotheses of Theorem 1.7. Examine the tables below to see that the containment of the projections and intersections of these proposed maximal subgroups are satisfied.

<table>
<thead>
<tr>
<th>$\widetilde{F_{11}}$</th>
<th>$F_{11}$</th>
</tr>
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<tbody>
<tr>
<td>$I_2 = F_1$</td>
<td>$I_1 = F_1$</td>
</tr>
<tr>
<td>$J_2 = 1$</td>
<td>$J_1 = Z$</td>
</tr>
<tr>
<td>$L_2 = F_1$</td>
<td>$L_1 = F_1$</td>
</tr>
<tr>
<td>$K_2 = 1$</td>
<td>$K_1 = Z$</td>
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</table>

<table>
<thead>
<tr>
<th>$\overrightarrow{F_{11}}$</th>
<th>$F_{11}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_2 = F_1$</td>
<td>$I_1 = F_1$</td>
</tr>
<tr>
<td>$J_2 = 1$</td>
<td>$J_1 = Z$</td>
</tr>
<tr>
<td>$L_2 = F_1$</td>
<td>$L_1 = F_1$</td>
</tr>
<tr>
<td>$K_2 = 1$</td>
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</table>

<table>
<thead>
<tr>
<th>$V$</th>
<th>$F_{11}$</th>
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<tbody>
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<td>$J_2 = Z$</td>
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<td>$L_2 = Z$</td>
<td>$L_1 = F_1$</td>
</tr>
<tr>
<td>$K_2 = Z$</td>
<td>$K_1 = Z$</td>
</tr>
</tbody>
</table>

Now, we must verify that parts $(ii)$, $(iii)$ and $(iv)$, of Theorem 1.7, are satisfied for each proposed maximal subgroup. Namely that $\left(\frac{I_2J_1}{J_1}\right)^{\sigma_1} = \frac{L_2K_1}{K_1}$, $\left(\frac{I_2 \cap J_1}{J_2}\right)^{\sigma_2} = \frac{L_2 \cap K_1}{K_2}$, and that the diagram commutes respectively. To verify parts $(ii)$ and $(iii)$, we must know $\sigma_1$ and $\sigma_2$. In our case, $\sigma_1 : \frac{F_1}{Z} \to \frac{F_1}{Z}$ is given $(xZ)^{\sigma_1} = xZ$. Observe that $\sigma_1$ will remain the same for each proposed maximal subgroup, and $\sigma_2$ changes. To verify part $(iv)$, that is, $\tilde{\sigma}_2 \circ \theta_1 = \theta_2 \circ \tilde{\sigma}_1$, for each proposed subgroup we will give the associated maps $\theta_1, \tilde{\sigma}_2, \tilde{\sigma}_1$ and $\theta_2$. Then we will verify that the diagram commutes. Recall that in verifying part $(iv)$, it is enough to check that the diagram commutes using the generators because our maps

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are linear. Let us now examine each of the proposed maximal subgroups, determine $\sigma_2$ and verify parts (ii), (iii) and (iv) of Theorem 1.7.

Consider $\overrightarrow{F_{11}}$. Then $\left(\frac{I_2 J_1}{J_1}\right)^{\sigma_1} = \left(\frac{F_1 Z}{Z}\right)^{\sigma_1} = \left(\frac{F_1}{Z}\right)^{\sigma_1} = \langle xZ \rangle^{\sigma_1} = \langle xZ \rangle$ and $\frac{L_2 K_1}{K_1} = \frac{F_1 Z}{Z} = \frac{F_1}{Z} = \langle xZ \rangle$. So, part (ii) is satisfied. Observe that $\sigma_2 : F_1 \rightarrow F_1$ is given by $(x)^{\sigma_2} = x$. Then $\left(\frac{I_2 \cap J_1}{J_2}\right)^{\sigma_2} = \left(\frac{F_1 \cap Z}{1}\right)^{\sigma_2} = (Z)^{\sigma_2} = Z$ and $\frac{L_2 \cap K_1}{K_2} = \frac{F_1 \cap Z}{1} = Z$. Hence, part (iii) is satisfied.

For $\overleftarrow{F_{11}}$, we have the following associated maps:

- $\theta_1 : \frac{F_1 Z}{Z} = \frac{F_1}{Z} \rightarrow \frac{F_1}{F_1 \cap Z} = \frac{F_1}{Z}$ is given by $(xZ)^{\theta_1} = xZ$;
- $\tilde{\sigma}_2 : \frac{F_1 \cap Z}{F_1} = \frac{F_1 \cap Z}{Z} \rightarrow \frac{F_1}{Z}$ is given by $(xZ)^{\tilde{\sigma}_2} = xZ$;
- $\tilde{\sigma}_1 : \frac{F_1 Z}{Z} = \frac{F_1}{Z} \rightarrow \frac{Z}{F_1} = \frac{Z}{Z}$ is given by $(xZ)^{\tilde{\sigma}_1} = xZ$;
- $\theta_2 : \frac{F_1 Z}{Z} = \frac{F_1}{Z} \rightarrow \frac{F_1 \cap Z}{F_1} = \frac{F_1}{Z}$ is given by $(xZ)^{\theta_2} = xZ$.

Since all of these maps are essentially the identity and defined in the same way, it is easy to see that the diagram commutes. Hence $\overleftarrow{F_{11}}$ is a maximal subgroup.

For $\overrightarrow{F_{11}}$, parts (ii), (iii) and (iv) are the same as $\overleftarrow{F_{11}}$, done above. The only difference is that $x$ gets sent to $x^{-1}$, but in $Q$, $xZ = x^{-1}Z$. Hence, parts (ii), (iii) and (iv) are satisfied.

Consider $V$. Then $\left(\frac{I_2 J_1}{J_1}\right)^{\sigma_1} = \left(\frac{Z}{Z}\right)^{\sigma_1} = \left(\frac{Z}{Z}\right)^{\sigma_1} = 1^{\sigma_1} = 1$ and $\frac{L_2 K_1}{K_1} = \frac{Z}{Z} = \frac{Z}{Z} = 1$. So, part (ii) is satisfied. Observe that $\sigma_2$ in this case is the identity map. Hence, part (iii) is obviously satisfied. Part (iv) is also satisfied because, for $V$, all of the maps are the identity.

Therefore the hypotheses of Theorem 1.7 are satisfied for all of our proposed maximal subgroups. Hence the maximal subgroups of $F_{11}$ are $\overleftarrow{F_{11}}, \overrightarrow{F_{11}}$ and $V$.

One can show that $F_{ij}$ are isomorphic, where $1 \leq i, j \leq 3$, and one can determine the maximal subgroups of the isomorphic groups by applying Theorem 1.8. Namely, one can use an automorphism of $Q \times Q$ to show that the subgroups are isomorphic. Then one uses this same automorphism to determine its maximal subgroups.

Let us now show that $F_{11} \cong F_{23}$. First it suffices to make the following observation: for $F_{11}$, $\sigma_1 : \frac{F_1}{Z} \rightarrow \frac{F_1}{Z}$ is given by $(xZ)^{\sigma_1} = xZ$, and for $F_{23}$, $\sigma_1 : \frac{F_2}{Z} \rightarrow \frac{F_3}{Z}$ is given by $(yZ)^{\sigma_1} = xyZ$. So, to show these two groups are isomorphic we need two automorphisms of $Q$. We need one to send $x$ to $y$ and the other to send $x$ to $xy$. Referring back to the 24
automorphisms of $Q$ listed in the introduction, we can use $\tau_9$ and $\tau_{17}$ respectively. Then $(\tau_9, \tau_{17}) \in Aut(Q \times Q)$, and $(F_{11})^{(\tau_9, \tau_{17})} = F_{23}$. Therefore these two groups are isomorphic.

To determine the maximal subgroups of $F_{23}$, we can examine the maximal subgroups of $F_{11}$ under $(\tau_9, \tau_{17})$. Let us now determine the maximal subgroups of $F_{23}$. Consider $\tilde{F_{11}}$. Then $((x, x))^{(\tau_9, \tau_{17})} = (\tilde{F_{23}})$. Consider $\tilde{F_{11}}$. Then since $\tilde{F_{11}}$ indicates we are using the automorphism that sends $x$ to $x^{-1}$, we know that $(\tilde{F_{11}})^{(\tau_9, \tau_{17})} = \tilde{F_{23}}$.

Consider $V$. Since $V$ has trivial sections and there are no other isomorphic copies of $V_4$, we know $V$ is definitely a maximal subgroup. Therefore, the maximal subgroups of $F_{23}$ are $\tilde{F_{23}}, \tilde{F_{23}}$ and $V$.

Similarly, one can show the remaining $F_{ij}$'s are isomorphic and determine their maximal subgroups. A list of the $F_{ij}$ and their maximal subgroups can be found in the table below.

<table>
<thead>
<tr>
<th>Groups</th>
<th>Maximal Subgroups</th>
<th>Groups</th>
<th>Maximal Subgroups</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_{11}$</td>
<td>$\tilde{F_{11}}, \tilde{F_{11}}, V$</td>
<td>$F_{23}$</td>
<td>$\tilde{F_{23}}, \tilde{F_{23}}, V$</td>
</tr>
<tr>
<td>$F_{12}$</td>
<td>$\tilde{F_{12}}, \tilde{F_{12}}, V$</td>
<td>$F_{31}$</td>
<td>$\tilde{F_{31}}, \tilde{F_{31}}, V$</td>
</tr>
<tr>
<td>$F_{13}$</td>
<td>$\tilde{F_{13}}, \tilde{F_{13}}, V$</td>
<td>$F_{32}$</td>
<td>$\tilde{F_{32}}, \tilde{F_{32}}, V$</td>
</tr>
<tr>
<td>$F_{21}$</td>
<td>$\tilde{F_{21}}, \tilde{F_{21}}, V$</td>
<td>$F_{33}$</td>
<td>$\tilde{F_{33}}, \tilde{F_{33}}, V$</td>
</tr>
<tr>
<td>$F_{22}$</td>
<td>$\tilde{F_{22}}, \tilde{F_{22}}, V$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Consider $\Delta_n$, $1 \leq n \leq 24$. These groups are determined by $\tau_n$, where $\tau_n \in Aut(Q)$. Recall that $\tau_n$ are defined and given in Section 0.1. Observe that $\Delta_n \leq Q \times Q$, and for any $\Delta_n$, its corresponding projection map (introduced in Goursat’s Theorem), is $\pi : Q \times Q \rightarrow Q$.

Consider $\Delta_n = \{(a, a^{\tau_n}) | a \in Q\}$. Then $\pi|_{\Delta_n} : \Delta_n \rightarrow Q$ is an isomorphism because it is a restriction of $\pi$ and because $Ker(\pi|_{\Delta_n}) = 1$. Hence $\Delta_n \cong Q$.

To understand the following table, it suffices to recall that each $\Delta_n$ is defined by its corresponding $\tau_n$, where $1 \leq n \leq 24$ and $\tau_n$ are listed in the Introduction. Considering inverse images and using the appropriate $\tau_n$, one determines the 3 maximal subgroups of $\Delta_n$ are $\langle (x, x^{\tau_n}) \rangle, \langle (y, y^{\tau_n}) \rangle$ and $\langle (xy, xy^{\tau_n}) \rangle$. A table which lists the maximal subgroups of the $\Delta_n$, $1 \leq n \leq 24$ is given below.
The groups of order 16 which appear in this lattice will be copies of $Q \times C_2$, $C_2 \times Q$, $C_4 \times C_4$ or $C_4 \times C_4$. The table that follows lists the groups of order 16 and their corresponding...
group structure. For these groups $1 \leq i, j \leq 3$ and $1 \leq p \leq 6$.

<table>
<thead>
<tr>
<th>Notation</th>
<th>I, J</th>
<th>I/J</th>
<th>Aut(I/J)</th>
<th>L, K</th>
<th>No. of subgroups</th>
<th>group structure</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{Q}_1$</td>
<td>$Q, Q$</td>
<td>1</td>
<td>1</td>
<td>$Z, Z$</td>
<td>1</td>
<td>$Q \times C_2$</td>
</tr>
<tr>
<td>$\hat{Q}_2$</td>
<td>$Z, Z$</td>
<td>1</td>
<td>1</td>
<td>$Q, Q$</td>
<td>1</td>
<td>$C_2 \times Q$</td>
</tr>
<tr>
<td>$\hat{F}_{ij}$</td>
<td>$F_i, F_i$</td>
<td>1</td>
<td>1</td>
<td>$F_j, F_j$</td>
<td>9</td>
<td>$C_4 \times C_4$</td>
</tr>
<tr>
<td>$\hat{M}_{ij}$</td>
<td>$Q, F_i$</td>
<td>$C_2$</td>
<td>1</td>
<td>$F_j, Z$</td>
<td>9</td>
<td>$C_4 \times C_4$</td>
</tr>
<tr>
<td>$\hat{M}^*_{ij}$</td>
<td>$F_i, Z$</td>
<td>$C_2$</td>
<td>1</td>
<td>$Q, F_j$</td>
<td>9</td>
<td>$C_4 \times C_4$</td>
</tr>
<tr>
<td>$B_p$</td>
<td>$Q, Z$</td>
<td>$V_4$</td>
<td>$S_3$</td>
<td>$Q, Z$</td>
<td>6</td>
<td>$Q \times C_2$</td>
</tr>
</tbody>
</table>

Now, let us begin to determine the maximal subgroups of the groups of order 16. Using the table above, we will begin by determining the maximal subgroups of the groups that have trivial sections, and then we will determine the maximal subgroups of the groups that have nontrivial sections.

Consider $\hat{Q}_1$, which is clearly isomorphic to $Q \times C_2$. Observe that this group has 3 generators, and because of this, Consequence 2 implies this group has 7 maximal subgroups. We claim the 7 maximal subgroups are $\hat{Q}_1$, $\hat{Y}_i$ and $\hat{W}_i$, where $1 \leq i \leq 3$. To determine that these are subgroups, we must verify the hypotheses of Theorem 1.7. Examine the tables below to see that the containment of the projections and intersections of these proposed maximal subgroups are satisfied.

<table>
<thead>
<tr>
<th>$\hat{Q}_1$</th>
<th>$\hat{Q}_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_2 = Q$</td>
<td>$I_1 = Q$</td>
</tr>
<tr>
<td>$J_2 = Q$</td>
<td>$J_1 = Q$</td>
</tr>
<tr>
<td>$L_2 = 1$</td>
<td>$L_1 = Z$</td>
</tr>
<tr>
<td>$K_2 = 1$</td>
<td>$K_1 = Z$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\hat{Y}_i$</th>
<th>$\hat{Q}_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_2 = F_i$</td>
<td>$I_1 = Q$</td>
</tr>
<tr>
<td>$J_2 = F_i$</td>
<td>$J_1 = Q$</td>
</tr>
<tr>
<td>$L_2 = Z$</td>
<td>$L_1 = Z$</td>
</tr>
<tr>
<td>$K_2 = Z$</td>
<td>$K_1 = Z$</td>
</tr>
</tbody>
</table>
Similarly, one can show the maximal subgroups of \( \hat{Q}_1 \).

Therefore, the hypotheses of Theorem 1 are satisfied. Observe that \( \sigma_2 \) changes for each subgroup that we are considering. Hence, it suffices to examine each of the proposed maximal subgroups, determine \( \sigma_2 \) and verify part (iii) of Theorem 1.

For \( \hat{Q}_1 \) and \( \hat{Y}_i \), 1 \( \leq \) \( i \) \( \leq \) 3, the associated \( \sigma_2 \) is the identity map. Hence, part (iii) is obviously satisfied.

For \( \hat{W}_i \), 1 \( \leq \) \( i \) \( \leq \) 3, \( \sigma_2 \) will change for each \( i \). So, we will consider each \( \hat{W}_i \) individually and verify part (iii) of Theorem 1 in each case. For \( \hat{W}_1 \), \( \sigma_2 : Q/I \to Z/I \) is given by \((yF_1)^{\sigma_2} = z\). Then \( \left( \frac{I_2 \cap J_1}{J_2} \right)^{\sigma_2} = \left( \frac{Q \cap Q}{F_1} \right)^{\sigma_2} = \left( \frac{Q}{F_1} \right)^{\sigma_2} = \langle z \rangle = Z \) and \( \frac{L_2 \cap K_1}{K_2} = \frac{Z \cap Z}{1} = Z \). Hence, part (iii) is satisfied.

For \( \hat{W}_2 \), \( \sigma_2 : Q/F_2 \to Z/I \) is given by \((xF_2)^{\sigma_2} = z\). Then \( \left( \frac{I_2 \cap J_1}{J_2} \right)^{\sigma_2} = \left( \frac{Q \cap Q}{F_2} \right)^{\sigma_2} = \left( \frac{Q}{F_2} \right)^{\sigma_2} = \langle z \rangle = Z \) and \( \frac{L_2 \cap K_1}{K_2} = \frac{Z \cap Z}{1} = Z \). Hence, part (iii) is satisfied.

For \( \hat{W}_3 \), \( \sigma_2 : Q/F_3 \to Z/I \) is given by \((yF_3)^{\sigma_2} = z\). Then \( \left( \frac{I_2 \cap J_1}{J_2} \right)^{\sigma_2} = \left( \frac{Q \cap Q}{F_3} \right)^{\sigma_2} = \left( \frac{Q}{F_3} \right)^{\sigma_2} = \langle z \rangle = Z \) and \( \frac{L_2 \cap K_1}{K_2} = \frac{Z \cap Z}{1} = Z \). Hence, part (iii) is satisfied.

Showing that the diagram commutes is trivial because the sections which define \( \hat{Q}_1 \) are trivial. Therefore, the hypotheses of Theorem 1 are satisfied, and the maximal subgroups of \( \hat{Q}_1 \) are \( \hat{Q}_1 \), \( \hat{Y}_i \) and \( \hat{W}_i \), where 1 \( \leq \) \( i \) \( \leq \) 3.

Similarly, one can show the maximal subgroups of \( \hat{Q}_2 \), an isomorphic copy of \( C_2 \times Q \), are \( \hat{Q}_1 \), \( \hat{Y}_j \) and \( \hat{W}_j \), where 1 \( \leq \) \( j \) \( \leq \) 3. A table with the maximal subgroups of \( \hat{Q}_1 \) and \( \hat{Q}_2 \) is given below.

<table>
<thead>
<tr>
<th>( \hat{W}_i )</th>
<th>( \hat{Q}_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( I_2 = Q )</td>
<td>( I_1 = Q )</td>
</tr>
<tr>
<td>( J_2 = F_i )</td>
<td>( J_1 = Q )</td>
</tr>
<tr>
<td>( L_2 = Z )</td>
<td>( L_1 = Z )</td>
</tr>
<tr>
<td>( K_2 = 1 )</td>
<td>( K_1 = Z )</td>
</tr>
</tbody>
</table>
Consider $\hat{F}_{12}$, which is clearly isomorphic to $C_4 \times C_4$. Because this group has 2 generators, Consequence 2 implies it has 3 maximal subgroups. We claim these maximal subgroups are $\hat{Y}_1$, $\hat{Y}_2$ and $F_{12}$. To determine that these are subgroups, we must verify the hypotheses of Theorem 1.7. Examine the tables below to see that the containment of the projections and intersections of these proposed maximal subgroups are satisfied.

<table>
<thead>
<tr>
<th>Groups</th>
<th>Maximal Subgroups</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{Q}_1$</td>
<td>$\hat{Q}_1$, $\hat{W}_1$, $\hat{W}_2$, $\hat{W}_3$, $\hat{Y}_1$, $\hat{Y}_2$, $\hat{Y}_3$</td>
</tr>
<tr>
<td>$\hat{Q}_2$</td>
<td>$\hat{Q}_2$, $\hat{W}_1$, $\hat{W}_2$, $\hat{W}_3$, $\hat{Y}_1$, $\hat{Y}_2$, $\hat{Y}_3$</td>
</tr>
</tbody>
</table>

Now, we must show that parts (ii) and (iii) of Theorem 1.7, namely $\left( \frac{I_2 J_1}{J_2} \right)^{\sigma_1} = \frac{L_2 K_1}{K_2}$ and $\left( \frac{I_2 \cap J_1}{J_2} \right)^{\sigma_2} = \frac{L_2 \cap K_1}{K_2}$ respectively, are satisfied for each proposed maximal subgroup. In order to do this, we must know $\sigma_1$ and $\sigma_2$. In our case, $\sigma_1 : \frac{F_1}{F_1} \to \frac{F_2}{F_2}$, which is the identity, and verifying part (ii) of Theorem 1.7 is trivial. Observe that $\sigma_2$ changes for each subgroup that we are considering. Hence, it suffices to examine each of the proposed maximal subgroups, determine $\sigma_2$ and verify part (iii) of Theorem 1.7.
For $Y_1$ and $Y_2$, the associated $\sigma_2$ is the identity map. Hence, part $(iii)$ is obviously satisfied.

For $F_{12}$, $\sigma_2 : \frac{F_1}{Z} \to \frac{F_2}{Z}$ is given by $(xZ)^{\sigma_2} = yZ$. Then \[
\left(\frac{I_2 \cap J_1}{J_2}\right)^{\sigma_2} = \left(\frac{F_1 \cap F_1}{Z}\right)^{\sigma_2} = \left(\frac{F_1}{Z}\right)^{\sigma_2} = \langle xZ \rangle^{\sigma_2} = \langle yZ \rangle = Z \text{ and } \frac{L_2 \cap K_1}{K_2} = \frac{F_2 \cap F_2}{Z} = \frac{F_2}{Z} = \langle yZ \rangle.\]

Hence, part $(iii)$ is satisfied.

Showing that the diagram commutes is trivial because the sections which define $\hat{F}_{12}$ are trivial. Therefore, the hypotheses of Theorem 1.7 are satisfied, and the maximal subgroups of $\hat{F}_{12}$ are $\hat{Y}_1$, $\hat{Y}_2$ and $F_{12}$.

One can show that $\hat{F}_{ij}$ are isomorphic, where $1 \leq i, j \leq 3$, and one can determine the maximal subgroups of the isomorphic groups by applying Theorem 1.8. Namely, one can use an automorphism of $Q \times Q$ to show that the subgroups are isomorphic. Then one uses this same automorphism to determine its maximal subgroups.

Let us now show that $\hat{F}_{12} \cong \hat{F}_{23}$. Notice that for these two groups, the associated $\sigma_1$ is the identity map. In order to show these groups are isomorphic, it suffices to recall $\tau_{13} \in Aut(Q)$, which is given by $x^{\tau_{13}} = y$ and $y^{\tau_{13}} = xy$. Then $(\tau_{13}, \tau_{13}) \in Aut(Q \times Q)$ and \[
\left(\hat{F}_{12}\right)^{(\tau_{13}, \tau_{13})} = \hat{F}_{23}.\]
Therefore these two groups are isomorphic.

To determine the maximal subgroups of $\hat{F}_{23}$, we can examine the maximal subgroups of $\hat{F}_{12}$ under $(\tau_{13}, \tau_{13})$. Let us now determine the maximal subgroups of $\hat{F}_{23}$. Consider $\hat{Y}_1$. Then \[
\left(\hat{Y}_1\right)^{(\tau_{13}, \tau_{13})} = \langle (x, x), (z, z) \rangle^{(\tau_{13}, \tau_{13})} = \langle (y, y), (z, z) \rangle = \hat{Y}_2.\]
Consider $\hat{Y}_2$. Then \[
\left(\hat{Y}_2\right)^{(\tau_{13}, \tau_{13})} = \langle (z, z), (y, y) \rangle^{(\tau_{13}, \tau_{13})} = \langle (z, z), (xy, xy) \rangle = \hat{Y}_3.\]
Consider $(\hat{F}_{12})^{(\tau_{13}, \tau_{13})} = \langle (xZ, yZ) \rangle^{(\tau_{13}, \tau_{13})} = \langle (yZ, xyZ) \rangle = \hat{F}_{23}$. Therefore, the maximal subgroups of $\hat{F}_{23}$ are $\hat{Y}_2$, $\hat{Y}_3$ and $F_{23}$.

Similarly, one can show the remaining $F_{ij}$’s are isomorphic and determine that their maximal subgroups are $Y_i$, $Y_j$ and $F_{ij}$. A list of the $F_{ij}$ and their maximal subgroups can be found in the table below.
Groups | Maximal Subgroups | Groups | Maximal Subgroups
--- | --- | --- | ---
\(\hat{F}_{11}\) | \(\hat{Y}_1, F_{11}, \hat{V}_1\) | \(\hat{F}_{23}\) | \(\hat{Y}_2, F_{23}, \hat{V}_3\)
\(\hat{F}_{12}\) | \(\hat{Y}_1, F_{12}, \hat{V}_2\) | \(F_{31}\) | \(\hat{Y}_3, F_{31}, \hat{V}_1\)
\(\hat{F}_{13}\) | \(\hat{Y}_1, F_{13}, \hat{V}_3\) | \(\hat{F}_{32}\) | \(\hat{Y}_3, F_{32}, \hat{V}_2\)
\(\hat{F}_{21}\) | \(\hat{Y}_2, F_{21}, \hat{V}_1\) | \(\hat{F}_{33}\) | \(\hat{Y}_3, F_{33}, \hat{V}_3\)
\(\hat{F}_{22}\) | \(\hat{Y}_2, F_{22}, \hat{V}_2\) |  |  

Before considering the remaining groups of order 16, it suffices to make an observation. Observe that \(C_4 \rtimes C_4 = \langle x, y \mid x^4 = y^4 = 1, x^y = x^{-1} \rangle\). Because \(C_4 \rtimes C_4\) is noncyclic and has a minimal generating set of two elements, Consequence 2 implies it contains 3 maximal subgroups. Although the group \(C_4 \rtimes C_4\) is not crucial to this dissertation, the number of generators it has is crucial.

Consider \(\hat{M}_{22} = \{(a, b) \mid a \in Q, b \in F_2, (aF_2)^{\sigma_1} = bZ\}\), where \(\sigma_1 : \frac{Q}{F_2} \to \frac{F_2}{Z}\) is given by \((xF_2)^{\sigma_1} = yZ\). So, \((y, 1), (x, y) \in \hat{M}_{22}, (y, 1)^4 = (x, y)^4 = 1, \langle (y, 1) \rangle < \langle (y, 1), (x, y) \rangle\) and \((y, 1)^{(x, y)} = (y^2, 1) = (y^{-1}, 1)\). So, \(\hat{M}_{22} = \langle (y, 1), (x, y) \rangle\), and one can believe that the group structure of \(\hat{M}_{22}\) is \(C_4 \rtimes C_4\). Therefore, \(\hat{M}_{22}\) is not cyclic and can be generated by 2 elements. Hence, it has 3 maximal subgroups, and we propose that these maximal subgroups are \(\hat{Y}_2, F_{12}\) and \(F_{32}\). To determine that these are subgroups, we must verify the hypotheses of Theorem 1.7. Examine the tables below to see that the containment of the projections and intersections of these proposed maximal subgroups are satisfied.

<table>
<thead>
<tr>
<th>(\hat{Y}_2)</th>
<th>(\hat{M}_{22})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(I_2 = F_2)</td>
<td>(I_1 = Q)</td>
</tr>
<tr>
<td>(J_2 = F_2)</td>
<td>(J_1 = F_2)</td>
</tr>
<tr>
<td>(L_2 = Z)</td>
<td>(L_1 = F_2)</td>
</tr>
<tr>
<td>(K_2 = Z)</td>
<td>(K_1 = Z)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(F_{12})</th>
<th>(\hat{M}_{22})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(I_2 = F_1)</td>
<td>(I_1 = Q)</td>
</tr>
<tr>
<td>(J_2 = Z)</td>
<td>(J_1 = F_2)</td>
</tr>
<tr>
<td>(L_2 = F_2)</td>
<td>(L_1 = F_2)</td>
</tr>
<tr>
<td>(K_2 = Z)</td>
<td>(K_1 = Z)</td>
</tr>
</tbody>
</table>
Now, we must verify that parts (ii), (iii) and (iv), of Theorem 1.7, are satisfied for each proposed maximal subgroup. Namely that \( \left( \frac{I_2 J_1}{J_1} \right)^{\sigma_1} = \frac{L_2 K_1}{K_1} \), \( \left( \frac{I_2 \cap J_1}{J_2} \right)^{\sigma_2} = \frac{L_2 \cap K_1}{K_2} \), and that the diagram commutes respectively. To verify parts (ii) and (iii), we must know \( \sigma_1 \) and \( \sigma_2 \). In our case, \( \sigma_1 : \frac{Q}{F_2} \rightarrow \frac{F_2}{Z} \) is given by \((x F_2)^{\sigma_1} = y Z\). Observe that \( \sigma_1 \) will remain the same for each proposed maximal subgroup, and \( \sigma_2 \) changes. To verify part (iv), that is, \( \tilde{\sigma}_2 \circ \theta_1 = \theta_2 \circ \tilde{\sigma}_1 \), for each proposed subgroup we will give the associated maps \( \theta_1 \), \( \tilde{\sigma}_2 \), \( \tilde{\sigma}_1 \) and \( \theta_2 \). Then we will verify that the diagram commutes. Recall that in verifying part (iv), it is enough to check that the diagram commutes using the generators because our maps are linear. Let us now examine each of the proposed maximal subgroups, determine \( \sigma_2 \) and verify parts (ii), (iii) and (iv) of Theorem 1.7.

Consider \( \tilde{\sigma}_2 \). Then \( \left( \frac{I_2 J_1}{J_1} \right)^{\sigma_1} = \left( \frac{F_2 F_2}{F_2} \right)^{\sigma_1} = \left( \frac{F_2}{F_2} \right)^{\sigma_1} = 1 \) and \( \frac{L_2 K_1}{K_1} = \frac{Z Z}{Z} = Z = 1 \). So, part (ii) is satisfied. The associated maps for \( \tilde{\sigma}_2 \) is the identity. Hence part (iii) is obviously satisfied.

For \( \tilde{\sigma}_1 \), the associated maps are \( \theta_1 : \frac{F_2 F_2}{F_2} \rightarrow \frac{F_2}{F_2 \cap F_2} \), \( \tilde{\sigma}_2 : \frac{F_2}{F_2 \cap F_2} \rightarrow \frac{Z}{Z \cap Z} \), and \( \tilde{\sigma}_1 : \frac{F_2 F_2}{F_2} \rightarrow \frac{Z Z}{Z} \), and \( \theta_2 : \frac{Z Z}{Z} \rightarrow \frac{Z}{Z \cap Z} \). Notice that all of these maps are the identity. Hence, the diagram commutes, and \( \tilde{\sigma}_1 \) is a maximal subgroup.

Consider \( F_{12} \). Then \( \left( \frac{I_2 J_1}{J_1} \right)^{\sigma_1} = \left( \frac{F_1 F_2}{F_2} \right)^{\sigma_1} = \left( \frac{F_2}{F_2} \right)^{\sigma_1} = (y F_2)^{\sigma_1} = (x F_2)^{\sigma_1} = (y Z) \) and \( \frac{L_2 K_1}{K_1} = \frac{F_2 Z}{Z} = \frac{F_2}{F_2} = \frac{y Z}{Z} \). So, part (ii) is satisfied. Observe that \( \sigma_2 : \frac{Z Z}{Z} \rightarrow \frac{F_2}{F_2} \) is given by \((x Z)^{\sigma_2} = y Z\). Then \( \left( \frac{I_2 \cap J_1}{J_2} \right)^{\sigma_2} = \left( \frac{F_1 \cap F_2}{Z} \right)^{\sigma_2} = \left( \frac{Z}{Z} \right)^{\sigma_2} = 1 \) and \( \frac{L_2 \cap K_1}{K_2} = \frac{F_2 \cap Z}{Z} = Z = 1 \). Hence, part (iii) is satisfied.

For \( F_{12} \), the following associated maps are:

\[
\begin{align*}
\theta_1 : & \frac{F_1 F_2}{F_2} = \frac{Q}{F_2} \rightarrow \frac{F_1}{F_1 \cap F_2} = \frac{F_1}{Z} \text{ is given by } (x F_2)^{\theta_1} = x Z; \\
\tilde{\sigma}_2 : & \frac{F_1 \cap F_2}{Z} = \frac{F_1}{Z} \rightarrow \frac{F_2 \cap Z}{Z} = \frac{F_2}{Z} \text{ is given by } (x Z)^{\tilde{\sigma}_2} = y Z; \\
\tilde{\sigma}_1 : & \frac{F_1 F_2}{F_2} = \frac{Q}{F_2} \rightarrow \frac{F_2 Z}{Z} = \frac{F_2}{Z} \text{ is given by } (x F_2)^{\tilde{\sigma}_1} = y Z;
\end{align*}
\]
\[ \theta_2 : \frac{F_2Z}{Z} = \frac{F_2}{Z} \rightarrow \frac{F_2}{F_2 \cap Z} = \frac{F_2}{Z} \] is given by \((yZ)^{\theta_2} = yZ\).

Then \(\tilde{\sigma}_2(\theta_1(xF_2)) = \tilde{\sigma}_2(xZ) = yZ\) and \(\theta_2(\tilde{\sigma}_1(xF_2)) = \theta_2(yZ) = yZ\). Hence, the diagram commutes, and \(F_{12}\) is a maximal subgroup.

Consider \(F_{32}\). Then \(\left( \frac{I_2J_1}{J_1} \right)^{\sigma_1} = \left( \frac{F_3F_2}{F_2} \right)^{\sigma_1} = \left( \frac{Q}{F_2} \right)^{\sigma_1} = \langle xF_2 \rangle^{\sigma_1} = \langle yZ \rangle\) and \(\frac{L_2K_1}{K_1} = \frac{F_2Z}{Z} = \frac{F_2}{Z} = \langle yZ \rangle\). So, part (ii) is satisfied. Observe that \(\sigma_2 : \frac{F_3}{Z} \rightarrow \frac{F_2}{Z}\) is given by \((xyZ)^{\sigma_2} = yZ\). Then \(\left( \frac{I_2 \cap J_1}{J_2} \right)^{\sigma_2} = \left( \frac{F_3 \cap F_2}{Z} \right)^{\sigma_2} = \left( \frac{Z}{Z} \right)^{\sigma_2} = 1\) and \(\frac{L_2 \cap K_1}{K_2} = \frac{F_2 \cap Z}{Z} = \frac{Z}{Z} = 1\). Hence, part (iii) is satisfied.

For \(F_{32}\), the following associated maps are:

- \(\theta_1 : \frac{F_3F_2}{F_3} = \frac{Q}{F_2} \rightarrow \frac{F_3}{F_3 \cap F_2} = \frac{F_3}{Z}\) is given by \((xF_2)^{\theta_1} = x\ yZ\);
- \(\tilde{\sigma}_2 : \frac{F_3 \cap F_2}{Z} = \frac{F_3}{Z} \rightarrow \frac{F_2 \cap Z}{Z} = \frac{F_2}{Z}\) is given by \((xyZ)^{\tilde{\sigma}_2} = yZ\);
- \(\tilde{\sigma}_1 : \frac{F_3F_2}{F_2} = \frac{Q}{F_2} \rightarrow \frac{F_3}{F_3 \cap F_2} = \frac{F_3}{Z}\) is given by \((xF_2)^{\tilde{\sigma}_1} = yZ\);
- \(\theta_2 : \frac{F_2 \cap Z}{Z} = \frac{Z}{Z} \rightarrow \frac{F_2}{F_2 \cap Z} = \frac{F_2}{Z}\) given by \((yZ)^{\theta_2} = yZ\).

Then \(\tilde{\sigma}_2(\theta_1(xF_2)) = \tilde{\sigma}_2(xyZ) = yz\) and \(\theta_2(\tilde{\sigma}_1(xF_2)) = \theta_2(yZ) = yZ\). Hence, the diagram commutes, and \(F_{32}\) is a maximal subgroup.

Therefore, the hypotheses of Theorem 1.7 are satisfied for all of our proposed maximal subgroups. Hence the maximal subgroups of \(\hat{M}_{22}\) are \(\hat{Y}_2\), \(F_{12}\) and \(F_{32}\).

One can show that \(\hat{M}_{ij}\) are isomorphic, where \(1 \leq i, j \leq 3\), by applying Theorem 1.8. Namely, one can use an automorphism of \(Q \times Q\) to show that the subgroups are isomorphic.

Let us now show that \(\hat{M}_{22} \cong \hat{M}_{12}\). First it suffices to make the following observation: for \(\hat{M}_{22}\), \(\sigma_1 : \frac{Q}{F_2} \rightarrow \frac{F_2}{Z}\) is given by \((xF_2)^{\sigma_1} = yZ\), and for \(\hat{M}_{12}\), \(\sigma_1 : \frac{Q}{F_1} \rightarrow \frac{F_2}{Z}\) is given by \((yF_1)^{\sigma_1} = yZ\). So, to show these two groups are isomorphic, we need two automorphisms of \(Q\). One should send \(x\) to \(y\) and \(y\) to \(x\), and the other should send \(y\) to \(y\). Referring back to the 24 automorphisms of \(Q\) given in the introduction, we can use \(\tau_9\) and \(\tau_{17}\) respectively.

Then \((\tau_9, \tau_{17}) \in Aut(Q \times Q)\) and \(\left( \hat{M}_{22} \right)^{(\tau_9, \tau_{17})} = \hat{M}_{12}\). Therefore, these two groups are isomorphic.

To determine the maximal subgroups of \(\hat{M}_{12}\) we examine the maximal subgroups of \(\hat{M}_{22}\) under the automorphism \((\tau_9, \tau_{17})\). Let us now determine the maximal subgroups of \(\hat{M}_{12}\).

Consider \(\hat{Y}_2\). Then \(\left( \hat{Y}_2 \right)^{(\tau_9, \tau_{17})} = \langle (y, y), (z, z) \rangle^{(\tau_9, \tau_{17})} = \langle (x, x), (z, z) \rangle = \hat{Y}_1\). Consider \(F_{12}\). Then \(\left( F_{12} \right)^{(\tau_9, \tau_{17})} = \langle (xZ, yZ) \rangle^{(\tau_9, \tau_{17})} = \langle (yZ, yZ) \rangle = F_{22}\). Consider \(F_{32}\) and recall
that \( xZ = x^{-1}Z \) in \( Q \). Then \( (F_{32})^{(\tau_y, \tau_{17})} = \langle (x^{-1}yZ, yZ) \rangle^{(\tau_y, \tau_{17})} = \langle xyZ, yZ \rangle = F_{32} \). Therefore, the maximal subgroups of \( M_{12} \) are \( Y_2, F_{22} \) and \( F_{32} \).

Similarly, one can show the remaining \( M_{ij} \)'s are isomorphic and determine that their maximal subgroups are \( Y_i \) and \( F_{ij} \), where \( j \) is fixed, \( i \neq j \) and \( 1 \leq i, j \leq 3 \).

Analogously, we can produce a similar argument for \( M_{ij} \) by noticing that this group is defined by reversing the roles of \( I, J \) and \( L, K \) in \( M_{ij} \). Namely, in reversing the roles, we are using the fact that \( \rho : (a, b) \mapsto (b, a) \) is an automorphism of \( Q \times Q \). The maximal subgroups of \( M_{ij} \) are \( Y_i \) and \( F_{ij} \), where \( i \) is fixed, \( i \neq j \) and \( 1 \leq i, j \leq 3 \). A table with the maximal subgroups of \( M_{ij} \) and \( M_{ij} \) is given below.

<table>
<thead>
<tr>
<th>Groups</th>
<th>Maximal Subgroups</th>
<th>Groups</th>
<th>Maximal Subgroups</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \overrightarrow{M_{11}} )</td>
<td>( Y_1, F_{21}, F_{31} )</td>
<td>( \overrightarrow{M_{11}} )</td>
<td>( Y_1, F_{21}, F_{31} )</td>
</tr>
<tr>
<td>( \overrightarrow{M_{12}} )</td>
<td>( Y_1, F_{22}, F_{32} )</td>
<td>( \overrightarrow{M_{12}} )</td>
<td>( Y_1, F_{22}, F_{32} )</td>
</tr>
<tr>
<td>( \overrightarrow{M_{13}} )</td>
<td>( Y_1, F_{23}, F_{33} )</td>
<td>( \overrightarrow{M_{13}} )</td>
<td>( Y_1, F_{22}, F_{23} )</td>
</tr>
<tr>
<td>( \overrightarrow{M_{21}} )</td>
<td>( Y_2, F_{11}, F_{31} )</td>
<td>( \overrightarrow{M_{21}} )</td>
<td>( Y_2, F_{11}, F_{31} )</td>
</tr>
<tr>
<td>( \overrightarrow{M_{22}} )</td>
<td>( Y_2, F_{12}, F_{32} )</td>
<td>( \overrightarrow{M_{22}} )</td>
<td>( Y_2, F_{12}, F_{32} )</td>
</tr>
<tr>
<td>( \overrightarrow{M_{23}} )</td>
<td>( Y_2, F_{13}, F_{33} )</td>
<td>( \overrightarrow{M_{23}} )</td>
<td>( Y_2, F_{13}, F_{33} )</td>
</tr>
<tr>
<td>( \overrightarrow{M_{31}} )</td>
<td>( Y_3, F_{11}, F_{21} )</td>
<td>( \overrightarrow{M_{31}} )</td>
<td>( Y_3, F_{11}, F_{21} )</td>
</tr>
<tr>
<td>( \overrightarrow{M_{32}} )</td>
<td>( Y_3, F_{12}, F_{22} )</td>
<td>( \overrightarrow{M_{32}} )</td>
<td>( Y_3, F_{12}, F_{22} )</td>
</tr>
<tr>
<td>( \overrightarrow{M_{33}} )</td>
<td>( Y_3, F_{13}, F_{23} )</td>
<td>( \overrightarrow{M_{33}} )</td>
<td>( Y_3, F_{13}, F_{23} )</td>
</tr>
</tbody>
</table>

Consider \( B_p, 1 \leq p \leq 6 \). These groups are defined using the automorphisms, \( \beta_p \), of \( V_4 \) that were given in Section 0.2. Since the Frattini subgroup of a 2-group is generated by the squares, we know that \( \Phi(B_p) = \langle (z, z) \rangle \), where \( |(z, z)| = 2 \). Notice that for \( B_p \), \( \sigma_1 : \frac{Q}{Z} \to \frac{Q}{Z} \), where \( \sigma_1 \) is defined using the definition of the corresponding \( \beta_p \). Hence, \( \left| \frac{B_p}{\Phi(B_p)} \right| = \frac{|B_p|}{|\Phi(B_p)|} = \frac{16}{2} = 8 \). Therefore, Consequence 2 implies this group has 3 generators and, as a consequence, 7 maximal subgroups.

Consider \( B_1 \), which is given by the triple \( \left( \frac{Q}{Z}, \frac{Q}{Z}, \beta_1 \right) \), where \( x^{\beta_1} = x \) and \( y^{\beta_1} = y \). We propose that its maximal subgroups are \( \Delta_k \), where \( 1 \leq k \leq 4 \) and \( F_{ii} \), where \( 1 \leq i \leq 3 \). To determine that these are subgroups, we must verify the hypotheses of Theorem 1.7. We will show that \( \Delta_k, 1 \leq k \leq 4 \), satisfies the hypotheses first. Then we will show that \( F_{ii}, 1 \leq i \leq 3 \), satisfies the hypotheses. Examine the table below to see that the containment
of the projections and intersections of $\Delta_k$ is satisfied.

<table>
<thead>
<tr>
<th>$\Delta_k$</th>
<th>$B_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_2 = Q$</td>
<td>$I_1 = Q$</td>
</tr>
<tr>
<td>$J_2 = 1$</td>
<td>$J_1 = Z$</td>
</tr>
<tr>
<td>$L_2 = Q$</td>
<td>$L_1 = Q$</td>
</tr>
<tr>
<td>$K_2 = 1$</td>
<td>$K_1 = Z$</td>
</tr>
</tbody>
</table>

Before verifying the remaining hypotheses of Theorem 1.7, it suffices to make a few observations. First, because the corresponding isomorphism for $B_1$ is given by $x^{\beta_1} = x$ and $y^{\beta_1} = y$, in order for $\Delta_k$ to be a maximal subgroup of $B_1$, the $\tau_k$ which define $\Delta_k$ must fix $x$, $y$ and $z$ or must send these elements to their inverses. Second, it is alright to include the elements being sent to their inverses because, in $Q$, $xZ = x^{-1}Z$, $yZ = y^{-1}Z$ and $xyZ = (xy)^{-1}Z$.

Now, we must show that parts (ii) and (iii) of Theorem 1.7, namely $(\frac{I_2 \cap J_1}{J_2})^{\sigma_1} = (\frac{I_2}{J_1})^{\sigma_1} = (\frac{L_2 \cap K_1}{K_2})^{\sigma_2}$ and $(\frac{QZ}{Z})^{\sigma_1}$, which is given by $\langle xZ \rangle^{\sigma_1} = \langle xZ \rangle$, $\langle yZ \rangle^{\sigma_1} = \langle yZ \rangle$ and $\langle xyZ \rangle^{\sigma_1} = \langle xyZ \rangle$. Observe that $\sigma_2$ changes for each $k$. Hence, it suffices to examine each of the proposed maximal subgroups, determine $\sigma_2$ and verify parts (ii) and (iii) of Theorem 1.7.

Consider $\Delta_1$, and let us verify parts (ii) and (iii). For part (ii), $\left(\frac{I_2 J_1}{J_1}\right)^{\sigma_1} = \left(\frac{QZ}{Z}\right)^{\sigma_1} = \left(\frac{Q}{Z}\right)^{\sigma_1}$, which is given by $\langle xZ \rangle^{\sigma_1} = \langle xZ \rangle$, $\langle yZ \rangle^{\sigma_1} = \langle yZ \rangle$ and $\langle xyZ \rangle^{\sigma_1} = \langle xyZ \rangle$. Also, $\frac{L_2 K_1}{K_1} = \frac{QZ}{Z} = \frac{Q}{Z} = \langle xZ \rangle$, $\langle yZ \rangle$ or $\langle xZ \rangle$, respectively. So, part (ii) is satisfied. The associated $\sigma_2$ for $\Delta_1$ is the identity. Hence part (iii) is obviously satisfied.

We can show that $\Delta_2$, $\Delta_3$ and $\Delta_4$ also satisfy parts (ii) and (iii) by using a similar process as done above for $\Delta_1$ and using the second observation given above.

To show the diagram commutes for $\Delta_1$, observe that its associated maps are as follows:

| $\theta_1 : \frac{QZ}{Z} \to \frac{Q}{Z}$ |
| $\tilde{\sigma}_2 : \frac{QZ}{Z} \to \frac{Q}{Z}$ |
| $\tilde{\sigma}_1 : \frac{QZ}{Z} \to \frac{Q}{Z}$ |
| $\theta_2 : \frac{QZ}{Z} \to \frac{Q}{Z}$ |

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Then since all of these maps are essentially the identity map and since $\Delta_1$ fixes $x, y$ and $z$, this diagram commutes.

Similarly, using the second observation given above we can conclude that the diagram commutes for $\Delta_2, \Delta_3$ and $\Delta_4$. Therefore, the hypotheses of Theorem 1.7 are satisfied, and $\Delta_1, \Delta_2, \Delta_3$ and $\Delta_4$ are maximal subgroups of $B_1$.

Since $B_1$ has 7 maximal subgroups, we must show it has 3 more. Examine the table below to see that the containment of the projections and intersections for $F_{ii}$ are satisfied, where $1 \leq i \leq 3$.

<table>
<thead>
<tr>
<th>$F_{ii}$</th>
<th>$B_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_2 = F_i$</td>
<td>$I_1 = Q$</td>
</tr>
<tr>
<td>$J_2 = Z$</td>
<td>$J_1 = Z$</td>
</tr>
<tr>
<td>$L_2 = F_i$</td>
<td>$L_1 = Q$</td>
</tr>
<tr>
<td>$K_2 = Z$</td>
<td>$K_1 = Z$</td>
</tr>
</tbody>
</table>

Consider $F_{11}$, and let us now verify parts $(ii)$ and $(iii)$ of Theorem 1.7. For part $(ii)$, $(I_2 J_1)^{\sigma_1} = (F_1 Z)^{\sigma_1} = (xZ)^{\sigma_1}$ and $L_2 K_1 = F_1 Z = F_1 Z = (xZ)$. So, part $(ii)$ is satisfied. Observe that $\sigma_2 : F_1 Z \rightarrow F_1 Z$ is given by $(xZ)^{\sigma_2} = xZ$. Then $(I_2 J_1)^{\sigma_2} = (F_1 Z)^{\sigma_2} = 1$ and $L_2 K_1 = F_1 Z = F_1 Z = 1$. Hence, part $(iii)$ is satisfied.

To show the diagram commutes for $F_{11}$, observe that its associated maps are as follows:

$\theta_1 : \frac{F_1 Z}{Z} = \frac{F_1}{Z} \rightarrow \frac{F_1}{Z} = \frac{F_1}{Z}$

$\tilde{\sigma}_2 : \frac{F_1 Z}{Z} = \frac{F_1}{Z} \rightarrow \frac{F_1}{Z} = \frac{F_1}{Z}$

$\tilde{\sigma}_1 : \frac{F_1 Z}{Z} = \frac{F_1}{Z} \rightarrow \frac{F_1}{Z} = \frac{F_1}{Z}$

$\theta_2 : \frac{F_1 Z}{Z} = \frac{F_1}{Z} \rightarrow \frac{F_1}{Z} = \frac{F_1}{Z}$.

Then since all of these maps are essentially the identity and are defined by sending $xZ$ to $xZ$, the diagram commutes. Therefore, $F_{11}$ is a maximal subgroup of $B_1$.

Similarly, we can show that $F_{22}$ and $F_{33}$ are maximal subgroups. Therefore the maximal subgroups of $B_1$ are $\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_1, \Delta_2$, and $\Delta_3$.

One can show that $B_p$ are isomorphic, where $1 \leq p \leq 6$, by applying Theorem 1.8. Namely, one can use an automorphism of $Q \times Q$ to show that the subgroups are isomorphic. Let us now show that $B_1 \cong B_4$. For $B_1$, $\sigma_1 : \frac{Q}{Z} \rightarrow \frac{Q}{Z}$ is given by $(xZ)^{\sigma_1} = xZ$ and $(yZ)^{\sigma_1} = yZ$. 

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For $B_4$, $\sigma_1 : \frac{Q}{Z} \to \frac{Q}{Z}$ is given by $(xZ)^{\sigma_1} = yZ$ and $(yZ)^{\sigma_1} = xyZ$. So, to show these two groups are isomorphic, we need two automorphisms of $Q$. One should send $x$ to $x$ and $y$ to $y$. The other should send $x$ to $y$ and $y$ to $xy$. Referring back to the 24 automorphisms of $Q$ given in the introduction, we can use $\tau_1$ and $\tau_{13}$ respectively. Then $(\tau_1, \tau_{13}) \in Aut(Q \times Q)$ and $(B_1)^{(\tau_1, \tau_{13})} = B_4$. Therefore, these two groups are isomorphic.

To determine the maximal subgroups of $B_4$ we examine the maximal subgroups of $B_1$ under the automorphism $(\tau_1, \tau_{13})$. Let us now determine the maximal subgroups of $B_4$. Consider $\Delta_1$. Then $(\Delta_1)^{(\tau_1, \tau_{13})} = \langle (x, x), (y, y) \rangle^{(\tau_1, \tau_{13})} = \langle (x, y), (y, xy) \rangle = \Delta_{13}$. Consider $\Delta_2$. Then $(\Delta_2)^{(\tau_1, \tau_{13})} = \langle (x, x), (y, y^{-1}) \rangle^{(\tau_1, \tau_{13})} = \langle (x, y), (y, x^{-1}y) \rangle = \Delta_{15}$. Consider $\Delta_3$. Then $(\Delta_3)^{(\tau_1, \tau_{13})} = \langle (x, x^{-1}), (y, y) \rangle^{(\tau_1, \tau_{13})} = \langle (x, y^{-1}), (y, xy) \rangle = \Delta_{16}$. Consider $\Delta_4$. Then $(\Delta_4)^{(\tau_1, \tau_{13})} = \langle (x, x^{-1}), (y, y^{-1}) \rangle^{(\tau_1, \tau_{13})} = \langle (x, y^{-1}), (y, xy^{-1}) \rangle = \Delta_{14}$. Consider $F_{11}$. Then $(F_{11})^{(\tau_1, \tau_{13})} = \langle (x, x), (z, z) \rangle^{(\tau_1, \tau_{13})} = \langle (x, y), (z, z) \rangle = F_{12}$. Consider $F_{22}$. Then $(F_{22})^{(\tau_1, \tau_{13})} = \langle (y, y), (z, z) \rangle^{(\tau_1, \tau_{13})} = \langle (y, xy), (z, z) \rangle = F_{23}$. Consider $F_{33}$. Then $(F_{33})^{(\tau_1, \tau_{13})} = \langle (xy, xy), (z, z) \rangle^{(\tau_1, \tau_{13})} = \langle (xy, x), (z, z) \rangle = F_{31}$. Therefore, the maximal subgroups of $B_4$ are $\Delta_{13}$, $\Delta_{14}$, $\Delta_{15}$, $\Delta_{16}$, $F_{12}$, $F_{23}$ and $F_{31}$.

Similarly, one can show the remaining $B_p$'s are isomorphic and determine their maximal subgroups, where $1 \leq p \leq 6$. A table with the maximal subgroups of $B_p$ is given below.

<table>
<thead>
<tr>
<th>Groups</th>
<th>Maximal Subgroups</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_1$</td>
<td>$\Delta_1$, $\Delta_2$, $\Delta_3$, $\Delta_4$, $F_{11}$, $F_{22}$, $F_{33}$</td>
</tr>
<tr>
<td>$B_2$</td>
<td>$\Delta_5$, $\Delta_6$, $\Delta_7$, $\Delta_8$, $F_{11}$, $F_{23}$, $F_{32}$</td>
</tr>
<tr>
<td>$B_3$</td>
<td>$\Delta_9$, $\Delta_{10}$, $\Delta_{11}$, $\Delta_{12}$, $F_{12}$, $F_{21}$, $F_{33}$</td>
</tr>
<tr>
<td>$B_4$</td>
<td>$\Delta_{13}$, $\Delta_{14}$, $\Delta_{15}$, $\Delta_{16}$, $F_{12}$, $F_{23}$, $F_{31}$</td>
</tr>
<tr>
<td>$B_5$</td>
<td>$\Delta_{17}$, $\Delta_{18}$, $\Delta_{19}$, $\Delta_{20}$, $F_{13}$, $F_{22}$, $F_{31}$</td>
</tr>
<tr>
<td>$B_6$</td>
<td>$\Delta_{21}$, $\Delta_{22}$, $\Delta_{23}$, $\Delta_{24}$, $F_{13}$, $F_{21}$, $F_{32}$</td>
</tr>
</tbody>
</table>

2.5 The Subgroups of Order 32

The groups of order 32 which appear in this lattice will be copies of $Q \times C_4$, $C_4 \times Q$ or $Q \triangleleft Q$. $Q \triangleleft Q$ is called a subdirect product of $Q$ and $Q$ with amalgamated factor group $H$, where $H$ is isomorphic to a quotient of a set by a direct product of the kernels of $Q$. In our case, the kernels are isomorphic to $C_4$ (ala Goursat’s Theorem). For more information
on this notation, see [2] P. 73. The table that follows lists the groups of order 32 and their corresponding group structure. For these groups, $1 \leq i, j \leq 3$.

<table>
<thead>
<tr>
<th>Notation</th>
<th>$I, J$</th>
<th>$I/J$</th>
<th>$\text{Aut}(I/J)$</th>
<th>$L, K$</th>
<th>No. of subgroups</th>
<th>group structure</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\overleftarrow{A}_j$</td>
<td>$Q, Q$</td>
<td>1</td>
<td>1</td>
<td>$F_j, F_j$</td>
<td>3</td>
<td>$Q \times C_4$</td>
</tr>
<tr>
<td>$\overleftarrow{A}_i$</td>
<td>$F_i, F_i$</td>
<td>1</td>
<td>1</td>
<td>$Q, Q$</td>
<td>3</td>
<td>$C_4 \times Q$</td>
</tr>
<tr>
<td>$S_{ij}$</td>
<td>$Q, F_i$</td>
<td>$C_2$</td>
<td>1</td>
<td>$Q, F_j$</td>
<td>9</td>
<td>$Q \times Q$</td>
</tr>
</tbody>
</table>

Consider $\overleftarrow{A}_1$, which is clearly isomorphic to $Q \times C_4$. Observe that this group has 3 generators, and because of this, Theorem 0.1 implies it has 7 maximal subgroups. We propose these 7 maximal subgroups are $\overleftarrow{Q}_1$, $\overleftarrow{F}_{i1}$, $\overleftarrow{M}_{i1}$, where $1 \leq i \leq 3$. To determine that these are subgroups, we must verify the hypotheses of Theorem 1.7. Examine the tables below to see that the containments of the projections and intersections of the proposed maximal subgroups are satisfied.

<table>
<thead>
<tr>
<th>$\overleftarrow{Q}_1$</th>
<th>$\overleftarrow{A}_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_2 = Q$</td>
<td>$I_1 = Q$</td>
</tr>
<tr>
<td>$J_2 = Q$</td>
<td>$J_1 = Q$</td>
</tr>
<tr>
<td>$L_2 = Z$</td>
<td>$L_1 = F_1$</td>
</tr>
<tr>
<td>$K_2 = Z$</td>
<td>$K_1 = F_1$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\overleftarrow{F}_{i1}$</th>
<th>$\overleftarrow{A}_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_2 = F_i$</td>
<td>$I_1 = Q$</td>
</tr>
<tr>
<td>$J_2 = F_i$</td>
<td>$J_1 = Q$</td>
</tr>
<tr>
<td>$L_2 = F_1$</td>
<td>$L_1 = F_1$</td>
</tr>
<tr>
<td>$K_2 = F_1$</td>
<td>$K_1 = F_1$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\overleftarrow{M}_{i1}$</th>
<th>$\overleftarrow{A}_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_2 = Q$</td>
<td>$I_1 = Q$</td>
</tr>
<tr>
<td>$J_2 = F_i$</td>
<td>$J_1 = Q$</td>
</tr>
<tr>
<td>$L_2 = F_1$</td>
<td>$L_1 = F_1$</td>
</tr>
<tr>
<td>$K_2 = Z$</td>
<td>$K_1 = F_1$</td>
</tr>
</tbody>
</table>

Now, we must show that parts (ii) and (iii) of Theorem 1.7, namely \( \left( \frac{I_2J_1}{J_1} \right)^{\sigma_1} = \frac{L_2K_1}{K_1} \) and \( \left( \frac{I_2 \cap J_1}{J_2} \right)^{\sigma_2} = \frac{L_2 \cap K_1}{K_2} \) respectively, are satisfied for each proposed maximal subgroup. In
order to do this, we must know $\sigma_1$ and $\sigma_2$. In our case, $\sigma_1 : \frac{Q}{Q} \rightarrow \frac{F_1}{F_1}$, which is the identity, and so verifying part (ii) is trivial. Observe that $\sigma_2$ changes for each subgroup we consider. Hence, it suffices to examine each of the proposed maximal subgroups, determine $\sigma_2$ and verify part (iii) of Theorem 1.7.

For $\hat{Q}_1$ and $\hat{F}_{11}$, $1 \leq i \leq 3$, the associated $\sigma_2$ is the identity. Hence, part (iii) is obviously satisfied.

For $\hat{M}_{11}$, $\sigma_2$ will change for each $i$. However, $\frac{L_2 \cap K_1}{K_2} = \frac{F_1 \cap F_1}{Z} = \frac{F_1}{Z} = \langle xZ \rangle$ will not change. So, to complete verifying part (iii), we will consider each $\hat{M}_{11}$ individually and examine $\left( \frac{I_2 \cap J_1}{J_2} \right)^{\sigma_2}$. For $\hat{M}_{11}$, $\sigma_2 : \frac{Q}{F_1} \rightarrow \frac{F_1}{Z}$ and is given by $(yF_1)^{\sigma_2} = xZ$. Then $\left( \frac{I_2 \cap J_1}{J_2} \right)^{\sigma_2} = \left( \frac{Q \cap Q}{F_1} \right)^{\sigma_2} = \left( \frac{Q}{F_1} \right)^{\sigma_2} = \langle yF_1 \rangle^{\sigma_2} = \langle xZ \rangle$. For $\hat{M}_{21}$, $\sigma_2 : \frac{Q}{F_2} \rightarrow \frac{F_1}{Z}$ and is given by $(xF_2)^{\sigma_2} = xZ$. Then $\left( \frac{I_2 \cap J_1}{J_2} \right)^{\sigma_2} = \left( \frac{Q \cap Q}{F_2} \right)^{\sigma_2} = \left( \frac{Q}{F_2} \right)^{\sigma_2} = \langle xF_2 \rangle^{\sigma_2} = \langle xZ \rangle$.

For $\hat{M}_{31}$, $\sigma_2 : \frac{Q}{F_3} \rightarrow \frac{F_1}{Z}$ and is given by $(yF_3)^{\sigma_2} = xZ$. Then $\left( \frac{I_2 \cap J_1}{J_2} \right)^{\sigma_2} = \left( \frac{Q \cap Q}{F_3} \right)^{\sigma_2} = \left( \frac{Q}{F_3} \right)^{\sigma_2} = \langle yF_3 \rangle^{\sigma_2} = \langle xZ \rangle$. Hence, in all 3 cases, part (iii) is satisfied.

Showing that the diagram commutes is trivial because the sections which define $\hat{A}_1$ are trivial. Therefore, the hypotheses of Theorem 1.7 are satisfied, and the maximal subgroups of $\hat{A}_1$ are $\hat{Q}_1$, $\hat{F}_{11}$, $\hat{F}_{21}$, $\hat{F}_{31}$, $\hat{M}_{11}$, $\hat{M}_{21}$ and $\hat{M}_{31}$.

One can show that $\hat{A}_j$ are isomorphic, where $1 \leq j \leq 3$, by applying Theorem 1.8. Namely, one can use an automorphism of $Q \times Q$ to show that the subgroups are isomorphic. Let us now show that $\hat{A}_1 \cong \hat{A}_2$. The associated $\sigma_1$ for $\hat{A}_1$ and $\hat{A}_2$ is the identity. So, to show $\hat{A}_1 \cong \hat{A}_2$, we need to recall $\tau_0 \in Aut(Q)$ given by $(x)^{\tau_0} = y$. Then $(1, \tau_0) \in Aut(Q \times Q)$ and $\left( \hat{A}_1 \right)^{(1, \tau_0)} = \hat{A}_2$. Similarly, to show that $\hat{A}_1 \cong \hat{A}_3$, recall that $\tau_{17} \in Aut(Q)$ given by $(x)^{\tau_0} = xy$. Then $(1, \tau_{17}) \in Aut(Q \times Q)$ and $\left( \hat{A}_1 \right)^{(1, \tau_{17})} = \hat{A}_3$. Hence, $\hat{A}_j$ are isomorphic.

To determine the maximal subgroups of $\hat{A}_2$ and $\hat{A}_3$, we will use the same automorphisms of $Q \times Q$ from above that we used to show $\hat{A}_j$ were isomorphic. Then we will examine the maximal subgroups of $\hat{A}_1$ under these automorphisms to attain the maximal subgroups of the isomorphic subgroup we are considering. Recall that the maximal subgroups of $\hat{A}_1$ are $\hat{Q}_1$, $\hat{F}_{11}$, $\hat{F}_{21}$, $\hat{F}_{31}$, $\hat{M}_{11}$, $\hat{M}_{21}$ and $\hat{M}_{31}$.
Let us now determine the maximal subgroups of $\hat{A}_2$. Consider $\hat{Q}_1$. Then $(\hat{Q}_1)^{(1, \tau_9)} = \langle (x, 1), (y, 1), (1, z) \rangle^{(1, \tau_9)} = \langle (x, 1), (y, 1), (1, z) \rangle = \hat{Q}_1$. Consider $\hat{F}_{11}$. Then $(\hat{F}_{11})^{(1, \tau_9)} = \langle (x, 1), (1, x) \rangle^{(1, \tau_9)} = \langle (x, 1), (1, y) \rangle = \hat{F}_{12}$. Consider $\hat{F}_{21}$. Then $(\hat{F}_{21})^{(1, \tau_9)} = \langle (y, 1), (1, x) \rangle^{(1, \tau_9)} = \langle (y, 1), (1, y) \rangle = \hat{F}_{22}$. Consider $\hat{F}_{31}$. Then $(\hat{F}_{31})^{(1, \tau_9)} = \langle (xy, 1), (1, x) \rangle^{(1, \tau_9)} = \langle (xy, 1), (1, y) \rangle = \hat{F}_{32}$. Consider $\hat{M}_{11}$. It is easy to see that $(y, x)$ and $(x, z)$ generate $\hat{M}_{11}$. Then $(\hat{M}_{11})^{(1, \tau_9)} = \langle (y, x), (x, z) \rangle^{(1, \tau_9)} = \langle (y, y), (x, z) \rangle = \hat{M}_{12}$ since $(y, y)$ and $(x, z) \in \hat{M}_{12}$ and in a finite group under an automorphism, generating sets go to generating sets. Consider $\hat{M}_{21}$. Then $(\hat{M}_{21})^{(1, \tau_9)} = \langle (x, x), (y, z) \rangle^{(1, \tau_9)} = \langle (x, y), (y, z) \rangle = \hat{M}_{22}$. Consider $\hat{M}_{31}$. Then $(\hat{M}_{31})^{(1, \tau_9)} = \langle (y, x), (xy, z) \rangle^{(1, \tau_9)} = \langle (y, y), (xy, z) \rangle = \hat{M}_{32}$. Therefore, the maximal subgroups of $\hat{A}_2$ are $\hat{Q}_1, \hat{F}_{12}, \hat{F}_{22}, \hat{F}_{32}, \hat{M}_{12}, \hat{M}_{22}$ and $\hat{M}_{32}$. Similarly, using $(1, \tau_{17})$, one can show the maximal subgroups of $\hat{A}_3$ are $\hat{Q}_1, \hat{F}_{13}, \hat{F}_{23}, \hat{F}_{33}, \hat{M}_{13}, \hat{M}_{23}$ and $\hat{M}_{33}$.

Analogously, we can produce a similar argument for $\hat{A}_i$, $1 \leq i \leq 3$, by noticing that this group is defined by reversing the roles of $I, J$ and $L, K$ in $\hat{A}_j$. Namely, in reversing the roles, we are using the fact that $\gamma : (a, b) \mapsto (b, a)$ is an automorphism of $Q \times Q$. The maximal subgroups of $\hat{A}_i$ are $\hat{F}_{ij}, \hat{M}_{ij}$ and $\hat{Q}_2$, where $i$ is fixed and $1 \leq j \leq 3$. A table with the maximal subgroups of $\hat{A}_j$ and $\hat{A}_i$, where $1 \leq i, j \leq 3$, is given below.

<table>
<thead>
<tr>
<th>Groups</th>
<th>Maximal Subgroups</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{A}_1$</td>
<td>$\hat{F}<em>{11}, \hat{F}</em>{21}, \hat{F}<em>{31}, \hat{M}</em>{11}, \hat{M}<em>{21}, \hat{M}</em>{31}, \hat{Q}_1$</td>
</tr>
<tr>
<td>$\hat{A}_2$</td>
<td>$\hat{F}<em>{12}, \hat{F}</em>{22}, \hat{F}<em>{32}, \hat{M}</em>{12}, \hat{M}<em>{22}, \hat{M}</em>{32}, \hat{Q}_1$</td>
</tr>
<tr>
<td>$\hat{A}_3$</td>
<td>$\hat{F}<em>{13}, \hat{F}</em>{23}, \hat{F}<em>{33}, \hat{M}</em>{13}, \hat{M}<em>{23}, \hat{M}</em>{33}, \hat{Q}_1$</td>
</tr>
<tr>
<td>$\hat{A}_1$</td>
<td>$\hat{F}<em>{11}, \hat{F}</em>{12}, \hat{F}<em>{13}, \hat{M}</em>{11}, \hat{M}<em>{12}, \hat{M}</em>{13}, \hat{Q}_2$</td>
</tr>
<tr>
<td>$\hat{A}_2$</td>
<td>$\hat{F}<em>{21}, \hat{F}</em>{22}, \hat{F}<em>{23}, \hat{M}</em>{21}, \hat{M}<em>{22}, \hat{M}</em>{23}, \hat{Q}_2$</td>
</tr>
<tr>
<td>$\hat{A}_3$</td>
<td>$\hat{F}<em>{31}, \hat{F}</em>{32}, \hat{F}<em>{33}, \hat{M}</em>{31}, \hat{M}<em>{32}, \hat{M}</em>{33}, \hat{Q}_2$</td>
</tr>
</tbody>
</table>

Consider $S_{11}$. Before determining the maximal subgroups of $S_{11}$, we will show that $S_{11}$ has 3 generators.

**Claim 1.** $\Phi(S_{11}) = V$, where $V$ is the subgroup of order 4 that is isomorphic to $V_4$. As a consequence, $S_{11}$ has three generators. Then Consequence 2 of Burnside’s Basis Theorem will imply this group has 7 maximal subgroups.
Proof: Since the Frattini subgroup of a 2-group is generated by the squares, we know that \( \Phi(U) \leq V \), for all \( U \leq Q \times Q \) since \( a,b \in Q \) and \( (a,b)^2 = (a^2, b^2) \leq V \). Observe that \((x,1) \in S_{11}\). So, \((x,1)^2 = (z,1) \in \Phi(S_{11})\). Observe that \((1,x) \in S_{11}\). So, \((1,x)^2 = (1,z) \in \Phi(S_{11})\). Therefore, \( \Phi(S_{11}) = \langle (z,1) \rangle \). 

As a consequence \( \frac{|S_{11}|}{|\Phi(S_{11})|} = \frac{|S_{11}|}{V} = \frac{32}{4} = 8 \). Therefore, Burnside’s Basis Theorem implies this group has 3 generators and 7 maximal subgroups. This concludes the proof of this claim.

Now, we can determine the maximal subgroups contained in \( S_{11} \). We propose that its maximal subgroups are \( \hat{F}_{11}, \hat{M}_{12}, \hat{M}_{13}, \hat{M}_{21}, \hat{M}_{31}, B_1 \) and \( B_2 \). To determine that these are the maximal subgroups, we must verify the hypotheses of Theorem 1.7 for each proposed maximal subgroup.

Consider \( \hat{F}_{11} \). Examine the table below to see that the containment of the projections and intersections is satisfied.

<table>
<thead>
<tr>
<th>( \hat{F}_{11} )</th>
<th>( S_{11} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( I_2 = F_1 )</td>
<td>( I_1 = Q )</td>
</tr>
<tr>
<td>( J_2 = F_1 )</td>
<td>( J_1 = F_1 )</td>
</tr>
<tr>
<td>( L_2 = F_1 )</td>
<td>( L_1 = Q )</td>
</tr>
<tr>
<td>( K_2 = F_1 )</td>
<td>( K_1 = F_1 )</td>
</tr>
</tbody>
</table>

Now, we must verify that parts \((ii), (iii)\) and \((iv)\), of Theorem 1.7, are satisfied. Namely that \( \left( \frac{I_2J_1}{J_1} \right)^{\sigma_1} = \frac{L_2K_1}{K_1} \), \( \left( \frac{J_2 \cap J_1}{J_2} \right)^{\sigma_2} = \frac{L_2 \cap K_1}{K_2} \), and that the diagram commutes respectively.

To verify parts \((ii)\) and \((iii)\), we must know \( \sigma_1 \) and \( \sigma_2 \). In our case, \( \sigma_1 : \frac{Q}{F_1} \to \frac{Q}{F_1} \) is given by \( (yF_1)^{\sigma_1} = yF_1 \), and \( \sigma_2 \) is the identity. So, \( \left( \frac{J_2 \cap J_1}{J_1} \right)^{\sigma_1} = \left( \frac{F_1F_1}{F_1} \right)^{\sigma_1} = \left( \frac{F_1}{F_1} \right)^{\sigma_1} = 1 \) and \( \frac{L_2K_1}{K_1} = \frac{F_1F_1}{F_1} = \frac{F_1}{F_1} = 1 \). Hence part \((ii)\) is satisfied. Since \( \sigma_2 \) is the identity, part \((iii)\) is obviously satisfied.

To verify part \((iv)\), that is, \( \tilde{\sigma_2} \circ \theta_1 = \theta_2 \circ \tilde{\sigma_1} \), let us make an observation. Observe that \( \frac{L_2}{L_2 \cap K_1} = \frac{F_1}{F_1 \cap F_1} = 1 \). So, for this diagram to commute, all of the associated maps must be the identity. One can easily verify that this is indeed true. Hence the hypotheses of Theorem 1.7 are satisfied, and \( \hat{F}_{11} \) is a maximal subgroup of \( S_{11} \). Since there are 7 maximal subgroups, we need 6 more.
Consider $\hat{M}_{12}$. Examine the table below to see that the containment of the projections and intersections is satisfied.

<table>
<thead>
<tr>
<th>$\hat{M}_{12}$</th>
<th>$S_{11}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_2 = Q$</td>
<td>$I_1 = Q$</td>
</tr>
<tr>
<td>$J_2 = F_1$</td>
<td>$J_1 = F_1$</td>
</tr>
<tr>
<td>$L_2 = F_2$</td>
<td>$L_1 = Q$</td>
</tr>
<tr>
<td>$K_2 = Z$</td>
<td>$K_1 = F_1$</td>
</tr>
</tbody>
</table>

Now, we must verify that parts (ii), (iii) and (iv), of Theorem 1.7, are satisfied. Namely that

$$\left(\frac{I_2 J_1}{J_1}\right)^{\sigma_1}_{\tau_1} = \frac{L_2 K_1}{K_1}, \quad \left(\frac{I_2 \cap J_1}{J_2}\right)^{\sigma_2}_{\tau_2} = \frac{L_2 \cap K_1}{K_2},$$

and that the diagram commutes respectively.

To verify parts (ii) and (iii), we must know $\sigma_1$ and $\sigma_2$. In our case, $\sigma_1 : \frac{Q}{F_1} \rightarrow \frac{Q}{F_1}$ is given by $(yF_1)^{\sigma_1} = yF_1$, and $\sigma_2 : \frac{Q}{F_1} \rightarrow \frac{F_2}{Z}$ is given by $(yF_1)^{\sigma_2} = yZ$. So, $\left(\frac{I_2 J_1}{J_1}\right)^{\sigma_1}_{\tau_1} = \left(\frac{QF_1}{F_1}\right)^{\sigma_1} = \frac{Q}{F_1} = (yF_1)$ and $\frac{L_2 K_1}{K_1} = \frac{F_2 F_1}{F_1} = \frac{Q}{F_1} = (yF_1)$. Hence part (ii) is satisfied. For part (iii), $\left(\frac{I_2 \cap J_1}{J_2}\right)^{\sigma_2}_{\tau_2} = \left(\frac{Q \cap F_1}{F_1}\right)^{\sigma_2} = \left(\frac{F_1}{F_1}\right)^{\sigma_2} = 1$ and $\frac{L_2 \cap K_1}{K_2} = \frac{F_2 \cap F_1}{Z} = \frac{Z}{Z} = 1$. Hence part (iii) is satisfied.

To verify part (iv), that is, $\tilde{\sigma}_2 \circ \theta_1 = \theta_2 \circ \tilde{\sigma}_1$, observe that the associated maps for $\hat{M}_{12}$ are:

- $\theta_1 : \frac{QF_1}{F_1} = \frac{Q}{F_1} \rightarrow \frac{Q \cap F_1}{F_1} = \frac{Q}{F_1}$ is given by $(yF_1)^{\theta_1} = yF_1$;
- $\tilde{\sigma}_2 : \frac{Q \cap F_1}{F_1} = \frac{Q}{F_1} \rightarrow \frac{F_2 F_1}{F_1} = \frac{F_2}{Z}$ is given by $(yF_1)^{\tilde{\sigma}_2} = yZ$;
- $\tilde{\sigma}_1 : \frac{QF_1}{F_1} = \frac{Q}{F_1} \rightarrow \frac{F_2 F_1}{F_1} = \frac{Q}{F_1}$ is given by $(yF_1)^{\tilde{\sigma}_1} = yF_1$;
- $\theta_2 : \frac{F_2 F_1}{F_1} = \frac{Q}{F_1} \rightarrow \frac{F_2 \cap F_1}{F_1} = \frac{F_2}{Z}$ is given by $(yF_1)^{\theta_2} = yZ$.

Then $\tilde{\sigma}_2 (\theta_1 (yF_1)) = \tilde{\sigma}_2 (yF_1) = yZ$ and $\theta_2 (\tilde{\sigma}_1 (yF_1)) = \theta_2 (yF_1) = yZ$. Hence the diagram commutes, and the hypotheses of Theorem 1.7 are satisfied. Therefore, $\hat{M}_{12}$ is a maximal subgroup of $S_{11}$.

Similarly, one can show that $\hat{M}_{13}$ is a maximal subgroup of $S_{11}$. One can do this by replacing $F_2$ with $F_3$ in the argument for $\hat{M}_{12}$ given above. Now we need 4 more maximal subgroups.

Consider $\hat{M}_{21}$. Examine the table below to see that the containment of the projections and intersections is satisfied.
Consider a subgroup of $\tilde{S}_\theta$. Similarly, one can show that

$$\left(\frac{I_2J_1}{J_1}\right)^{\sigma_1} = \frac{L_2K_1}{K_1}, \quad \left(\frac{I_2 \cap J_1}{J_2}\right)^{\sigma_2} = \frac{L_2 \cap K_1}{K_2},$$

and that the diagram commutes respectively.

To verify parts (ii) and (iii), we must know $\sigma_1$ and $\sigma_2$. In our case, $\sigma_1 : \frac{Q}{F_1} \rightarrow \frac{Q}{F_1}$ is given by $(yF_1)^{\sigma_1} = yF_1$, and $\sigma_2 : \frac{F_2}{Z} \rightarrow \frac{Q}{F_1}$ is given by $(yZ)^{\sigma_2} = yF_1$. So, $\left(\frac{I_2J_1}{J_1}\right)^{\sigma_1} = \frac{F_2F_1}{F_1} = \left(\frac{Q}{F_1}\right)^{\sigma_1} = (yF_1)^{\sigma_1} = (yF_1)$ and $\frac{L_2K_1}{K_1} = \frac{QF_1}{F_1} = \frac{Q}{F_1} = \langle yF_1 \rangle$. Hence part (ii) is satisfied. For part (iii), $\left(\frac{I_2 \cap J_1}{J_2}\right)^{\sigma_2} = \left(\frac{F_2 \cap F_1}{Z}\right)^{\sigma_2} = \left(\frac{Z}{Z}\right)^{\sigma_2} = 1$ and $\frac{L_2 \cap K_1}{K_2} = \frac{Q \cap F_1}{F_1} = \frac{F_1}{F_1} = 1$. Hence part (iii) is satisfied.

To verify part (iv), that is, $\tilde{\sigma}_2 \circ \theta_1 = \theta_2 \circ \tilde{\sigma}_1$, observe that the associated maps for $\tilde{M}_{21}$ are:

- $\theta_1 : \frac{F_2}{F_1} = \frac{Q}{F_1} \rightarrow \frac{F_2 \cap F_1}{F_2} = \frac{F_2}{Z}$ is given by $(yF_1)^{\theta_1} = yZ$;
- $\tilde{\sigma}_2 : \frac{F_2 \cap F_1}{F_2} = \frac{Z}{F_2} \rightarrow \frac{Q \cap F_1}{Q} = \frac{Q}{F_1}$ is given by $(yZ)^{\tilde{\sigma}_2} = yF_1$;
- $\tilde{\sigma}_1 : \frac{Q}{F_1} = \frac{Q \cap F_1}{Q} \rightarrow \frac{QF_1}{F_1} = \frac{Q}{F_1}$ is given by $(yF_1)^{\tilde{\sigma}_1} = yF_1$;
- $\theta_2 : \frac{QF_1}{F_1} = \frac{Q \cap F_1}{Q} \rightarrow \frac{Q}{F_1}$ is given by $(yF_1)^{\theta_2} = yF_1$.

Then $\tilde{\sigma}_2 (\theta_1 (yF_1)) = \tilde{\sigma}_2 (yZ) = yF_1$ and $\theta_2 (\tilde{\sigma}_1 (yF_1)) = \theta_2 (yF_1) = yF_1$. Hence the diagram commutes, and the hypotheses of Theorem 1.7 are satisfied. Therefore, $\tilde{M}_{21}$ is a maximal subgroup of $S_{11}$.

Similarly, one can show that $\tilde{M}_{31}$ is a maximal subgroup of $S_{11}$. One can do this by replacing $F_2$ with $F_3$ in the argument for $\tilde{M}_{21}$ given above. Now we need 2 more maximal subgroups.

<table>
<thead>
<tr>
<th>$\tilde{M}_{21}$</th>
<th>$S_{11}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_2 = F_2$</td>
<td>$I_1 = Q$</td>
</tr>
<tr>
<td>$J_2 = Z$</td>
<td>$J_1 = F_1$</td>
</tr>
<tr>
<td>$L_2 = Q$</td>
<td>$L_1 = Q$</td>
</tr>
<tr>
<td>$K_2 = F_1$</td>
<td>$K_1 = F_1$</td>
</tr>
</tbody>
</table>

Consider $B_1$, which is given by the triple $\left(\frac{Q}{Z}, \frac{Q}{Z}, \beta_1\right)$, where $x^{\beta_1} = x$ and $y^{\beta_1} = y$. Let us now verify that this group satisfies the hypotheses of Theorem 1.7. Examine the table below to see that the containment of the projections and intersections is satisfied.
Similarly, one can show that the hypotheses of Theorem 1, are satisfied. Namely that
\[
\left( \frac{I_2 J_1}{J_1} \right)^{\sigma_1} = \frac{L_2 K_1}{K_1}, \quad \left( \frac{I_2 \cap J_1}{J_2} \right)^{\sigma_2} = \frac{L_2 \cap K_1}{K_2},
\]
and that the diagram commutes respectively.

To verify parts (ii) and (iii), we must know \( \sigma_1 \) and \( \sigma_2 \). In our case, \( \sigma_1 : Q \rightarrow F_1 \) is given by \((yF_1)^{\sigma_1} = yF_1 \), and \( \sigma_2 : Q \rightarrow Z \) is given by \((xZ)^{\sigma_2} = xZ \) and \((yZ)^{\sigma_2} = yZ \). Then
\[
\left( \frac{I_2 J_1}{J_1} \right)^{\sigma_1} = \left( \frac{QF_1}{F_1} \right)^{\sigma_1} = \left( \frac{Q}{F_1} \right)^{\sigma_1} = (yF_1)^{\sigma_1} = yF_1.
\]

Hence part (ii) is satisfied. For part (iii), \( \left( \frac{I_2 \cap J_1}{J_2} \right)^{\sigma_2} = \left( \frac{Q \cap F_1}{Z} \right)^{\sigma_2} = \left( \frac{F_1}{Z} \right)^{\sigma_2} = (xZ)^{\sigma_2} = (xZ) \) and \( \frac{L_2 \cap K_1}{K_2} = \frac{Q \cap F_1}{Z} = \frac{F_1}{Z} = (xZ) \). Hence part (iii) is satisfied.

To verify part (iv), that is, \( \tilde{\sigma}_2 \circ \theta_1 = \theta_2 \circ \tilde{\sigma}_1 \), observe that the associated maps for \( B_1 \) are:
\[
\theta_1 : \frac{QF_1}{F_1} = \frac{Q}{F_1} \rightarrow \frac{Q}{Q \cap F_1} = \frac{Q}{F_1} \text{ is given by } (yF_1)^{\theta_1} = yF_1;
\]
\[
\tilde{\sigma}_2 : \frac{Q \cap F_1}{F_1} = \frac{Q}{F_1} \rightarrow \frac{Q \cap F_1}{Q \cap F_1} = \frac{Q}{F_1} \text{ is given by } (yF_1)^{\tilde{\sigma}_2} = yF_1;
\]
\[
\tilde{\sigma}_1 : \frac{QF_1}{F_1} = \frac{Q}{F_1} \rightarrow \frac{QF_1}{Q} = \frac{Q}{F_1} \text{ is given by } (yF_1)^{\tilde{\sigma}_1} = yF_1;
\]
\[
\theta_2 : \frac{QF_1}{F_1} = \frac{Q}{F_1} \rightarrow \frac{Q \cap F_1}{Q \cap F_1} = \frac{Q}{F_1} \text{ is given by } (yF_1)^{\theta_2} = yF_1.
\]

Since all of these maps are essentially the identity, the diagram commutes. Hence, the hypotheses of Theorem 1 are satisfied, and \( B_1 \) is a maximal subgroup of \( S_{11} \).

Similarly, one can show that \( B_2 \) is a maximal subgroup of \( S_{11} \), where \( B_2 \) is given by the triple \( \left( \frac{Q}{Z}, \frac{Q}{Z}, \beta_2 \right) \) and \( x^{\beta_2} = x \) and \( y^{\beta_2} = xy \). Therefore, the maximal subgroups of \( S_{11} \) are
\[
\overline{F_{11}}, \overline{M_{12}}, \overline{M_{13}}, \overline{M_{21}}, \overline{M_{31}}, B_1 \text{ and } B_2.
\]

One can show that \( S_{ij} \) are isomorphic, where \( 1 \leq i, j \leq 3 \), by applying Theorem 1. Namely, one can use an automorphism of \( Q \times Q \) to show that the subgroups are isomorphic. Let us now show that \( S_{11} \cong S_{23} \). For \( S_{11} \), \( \sigma_1 : \frac{Q}{F_1} \rightarrow \frac{Q}{F_1} \) is given by \((yF_1)^{\sigma_1} = yF_1 \). For \( S_{23} \), \( \sigma_1 : \frac{Q}{F_3} \rightarrow \frac{Q}{F_3} \) is given by \((xF_2)^{\sigma_1} = yF_3 \). So, to show these two groups are isomorphic, we need two automorphisms of \( Q \). One should send \( x \) to \( y \) and \( y \) to \( x \). The other should send \( y \) to \( y \) and \( x \) to \( xy \). Referring back to the 24 automorphisms of \( Q \) given in the introduction,
we can use $\tau_9$ and $\tau_{17}$ respectively. Then $(\tau_9, \tau_{17}) \in \text{Aut}(Q \times Q)$ and $(S_{11})^{(\tau_9, \tau_{17})} = S_{23}$. Therefore, these two groups are isomorphic.

To determine the maximal subgroups of $S_{23}$ we examine the maximal subgroups of $S_{11}$ under the automorphism $(\tau_9, \tau_{17})$. Let us now determine the maximal subgroups of $S_{23}$. Consider $\overleftarrow{F_{11}}$. Then $\left(\overleftarrow{F_{11}}\right)^{(\tau_9, \tau_{17})} = \langle (x,1), (1,x) \rangle^{(\tau_9, \tau_{17})} = \langle (y,1), (1,xy) \rangle = \overleftarrow{F_{23}}$. Consider $\overleftarrow{M_{12}}$. Then $\left(\overleftarrow{M_{12}}\right)^{(\tau_9, \tau_{17})} = \langle (y,y), (x,z) \rangle^{(\tau_9, \tau_{17})} = \langle (x,y), (y,z) \rangle = \overleftarrow{M_{22}}$. Consider $\overleftarrow{M_{13}}$, and recall that $xZ = x^{-1}Z$ in $Q$. Then $\left(\overleftarrow{M_{13}}\right)^{(\tau_9, \tau_{17})} = \langle (y,xy), (x,z) \rangle^{(\tau_9, \tau_{17})} = \langle (x,x^{-1}), (y,z) \rangle = \overleftarrow{M_{21}}$. Consider $\overleftarrow{M_{21}}$. Then $\left(\overleftarrow{M_{21}}\right)^{(\tau_9, \tau_{17})} = \langle (y,y), (z,x) \rangle^{(\tau_9, \tau_{17})} = \langle (x,y), (z,xy) \rangle = \overleftarrow{M_{13}}$. Consider $\overleftarrow{M_{31}}$, and recall that $xyZ = x^{-1}yZ$ in $Q$. Then $\left(\overleftarrow{M_{31}}\right)^{(\tau_9, \tau_{17})} = \langle (xy,y), (z,x) \rangle^{(\tau_9, \tau_{17})} = \langle (x^{-1}y,y), (z,xy) \rangle = \overleftarrow{M_{33}}$. Consider $B_1$, the group whose corresponding isomorphism is given by $x^{\beta_1} = x$ and $y^{\beta_1} = y$. Then $(x,x)^{(\tau_9, \tau_{17})} = (y,xy)$ and $(y,xy)^{(\tau_9, \tau_{17})} = (x,x^{-1})$. Since we are in a finite group and $(\tau_9, \tau_{17})$ is an automorphism, it takes a generating set to a generating set. So, the group given by the triple whose corresponding isomorphism is the automorphism of $V_4$ which sends $y$ to $xy$ and $x$ to $y$ is the image of $B_1$ under $(\tau_9, \tau_{17})$. This automorphism is $\beta_4$. Hence $(B_4)^{(\tau_9, \tau_{17})} = B_4$. Consider $B_2$, the group whose corresponding isomorphism is given by $x^{\beta_2} = x$ and $y^{\beta_2} = xy$. Then $(x,x)^{(\tau_9, \tau_{17})} = (y,xy)$ and $(y,xy)^{(\tau_9, \tau_{17})} = (x,x^{-1})$. Again, since we are in a finite group and $(\tau_9, \tau_{17})$ is an automorphism, it takes a generating set to a generating set. So, the group given by the triple whose corresponding isomorphism is the automorphism of $V_4$ which sends $y$ to $xy$ and $x$ to $x$ is the image of $B_2$ under $(\tau_9, \tau_{17})$. Observe that we can use $x$ instead of $x^{-1}$ because in $Q$, $xZ = x^{-1}Z$. This automorphism is $\beta_2$. Hence $(B_2)^{(\tau_9, \tau_{17})} = B_2$. Therefore, the maximal subgroups of $S_{23}$ are $\overleftarrow{F_{23}}, \overleftarrow{M_{22}}, \overleftarrow{M_{21}}, \overleftarrow{M_{13}}, \overleftarrow{M_{33}}, B_4$ and $B_2$.

Similarly, one can show the remaining $S_{ij}$’s are isomorphic and determine that their maximal subgroups are $\overleftarrow{F_{ij}}, \overleftarrow{M_{ik}}$ where $i$ is fixed and $k \neq i$, and $\overleftarrow{M_{kj}}$ where $j$ is fixed and $k \neq j$, $1 \leq k \leq 3$. A table with the maximal subgroups of $S_{ij}$ is given below.
<table>
<thead>
<tr>
<th>Groups</th>
<th>Maximal Subgroups</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_{11}$</td>
<td>$F_{11}, B_1, B_2, \overrightarrow{M_{12}}, \overrightarrow{M_{13}}, \overrightarrow{M_{21}}, \overrightarrow{M_{31}}$</td>
</tr>
<tr>
<td>$S_{12}$</td>
<td>$F_{12}, B_3, B_4, \overrightarrow{M_{11}}, \overrightarrow{M_{13}}, \overrightarrow{M_{22}}, \overrightarrow{M_{32}}$</td>
</tr>
<tr>
<td>$S_{13}$</td>
<td>$F_{13}, B_5, B_6, \overrightarrow{M_{11}}, \overrightarrow{M_{12}}, \overrightarrow{M_{23}}, \overrightarrow{M_{33}}$</td>
</tr>
<tr>
<td>$S_{21}$</td>
<td>$F_{21}, B_3, B_6, \overrightarrow{M_{22}}, \overrightarrow{M_{23}}, \overrightarrow{M_{11}}, \overrightarrow{M_{31}}$</td>
</tr>
<tr>
<td>$S_{22}$</td>
<td>$F_{22}, B_1, B_5, \overrightarrow{M_{21}}, \overrightarrow{M_{23}}, \overrightarrow{M_{12}}, \overrightarrow{M_{32}}$</td>
</tr>
<tr>
<td>$S_{23}$</td>
<td>$F_{23}, B_2, B_4, \overrightarrow{M_{21}}, \overrightarrow{M_{22}}, \overrightarrow{M_{13}}, \overrightarrow{M_{33}}$</td>
</tr>
<tr>
<td>$S_{31}$</td>
<td>$F_{31}, B_4, B_5, \overrightarrow{M_{32}}, \overrightarrow{M_{33}}, \overrightarrow{M_{11}}, \overrightarrow{M_{21}}$</td>
</tr>
<tr>
<td>$S_{32}$</td>
<td>$F_{32}, B_2, B_6, \overrightarrow{M_{31}}, \overrightarrow{M_{33}}, \overrightarrow{M_{12}}, \overrightarrow{M_{22}}$</td>
</tr>
<tr>
<td>$S_{33}$</td>
<td>$F_{33}, B_1, B_3, \overrightarrow{M_{31}}, \overrightarrow{M_{32}}, \overrightarrow{M_{13}}, \overrightarrow{M_{23}}$</td>
</tr>
</tbody>
</table>
2.6 Subgroup Lattice of $Q \times Q$

In concluding this chapter, we use the construction given in Sections 2.1 - 2.5 to give the subgroup lattice of $Q \times Q$. We were able to construct this subgroup lattice and all the subgroup lattices in this dissertation using The Geometer’s Sketchpad Version 5 [1].
Chapter 3

Further Results about Constructions and Subgroup Lattices

In this chapter, we would like to discuss $Q \times D_8$ and give its subgroup lattice, where $D_8$ denotes the dihedral group of order 8. This group is of interest particularly because it contains an extraspecial group of order 32, and we would like to identify it. Section 3.1 will introduce $Q \times D_8$ and describe how we used Goursat’s theorem to calculate the number of subgroups. The details of the subgroup lattice construction of $Q \times D_8$ will not be included in this dissertation. As one can imagine, the details would be similar to the construction of $Q \times Q$. In Section 3.2, we will discuss extraspecial groups and give a classification of them. In Section 3.3, we will give the subgroup lattices of $Q \times Q$ and $Q \times D_8$. Then we will give the subgroup lattices of the two extraspecial groups of order 32 that were found using the subgroup lattices of $Q \times Q$ and $Q \times D_8$.

3.1 $Q \times D_8$

Before giving the subgroup lattice of $Q \times D_8$, let us introduce notation and show how one would count the number of subgroups it has. The quaternions will have the presentation $Q = \langle x, y | x^4 = y^4 = 1, x^2 = y^2 \rangle$. Let $F_1 = \langle x \rangle$, $F_2 = \langle y \rangle$, and $F_3 = \langle xy \rangle$ be the subgroups of order 4 in $Q$. Let the subgroup of order 2 be $Z_1 = \langle x^2 \rangle$. Let $\mathbf{E} = \{Q, F_i, Z_1, 1 | 1 \leq i \leq 3 \}$. The dihedral group of order 8 will have the presentation $D_8 = \langle r, s | r^4 = s^2 = 1, r^s = r^{-1} \rangle$. Let $G_1 = \langle r^2, r s \rangle$, $G_2 = \langle r^2, s \rangle$, and $G_3 = \langle r \rangle$ be the subgroups of order 4 in $D_8$. Let $H_1 = \langle r s \rangle$, $H_2 = \langle sr \rangle$, $H_3 = \langle sr^2 \rangle$, $H_4 = \langle s \rangle$, and $Z_2 = \langle r^2 \rangle$ be the subgroups of order 2 in $D_8$. Let $\mathbf{F} = \{G_i, H_l, Z_2 | 1 \leq i \leq 3, 1 \leq l \leq 4 \}$. The subgroup lattice of $D_8$ can be found below.
Observe that our aim is to determine the number of subgroups in \( Q \times D_8 \) and the orders of the respective subgroups. To do this we use Goursat’s theorem to examine the quotients \( I/J \) and \( L/K \), where \( I, J \in E \) and \( L, K \in F \). For a reference on counting subgroups, see [10].

In order to calculate the orders of the subgroups we will use the following: If \( U \leq A \times B \), then \( |U| = |I/L| |I/J| \).

In the following 4 cases, the number of subgroups and their respective orders will be determined. For the notation used, \( 1 \leq i, k \leq 3 \), \( 1 \leq l \leq 4 \), and \( 1 \leq j \leq 2 \). If \( j = 1 \), then \( t = 1 \) or 2. If \( j = 2 \), then \( t = 3 \) or 4. (i.e., \( t \) depends on \( j \)) Also note that the choice of \( I \) and \( J \) will come from \( E \) and the choice of \( L \) and \( K \) will come from \( F \).

**Case 1.** Let \( I = Q \), and let \( J \in E \). Then

<table>
<thead>
<tr>
<th>( I, J )</th>
<th>( I/J )</th>
<th>( \text{Aut}(I/J) )</th>
<th>( L, K )</th>
<th>No. of subgroups</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Q, Q )</td>
<td>1</td>
<td>( D_8, D_8 )</td>
<td>1</td>
<td>of order 64</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( G_k, G_k )</td>
<td>3</td>
<td>of order 32</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( H_l, H_t )</td>
<td>4</td>
<td>of order 16</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( Z_2, Z_2 )</td>
<td>1</td>
<td>of order 16</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( 1, 1 )</td>
<td>1</td>
<td>of order 8</td>
</tr>
<tr>
<td>( Q, F_i )</td>
<td>( C_2 )</td>
<td>( D_8, G_k )</td>
<td>9</td>
<td>of order 32</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( G_j, H_t )</td>
<td>12</td>
<td>of order 16</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( G_k, Z_2 )</td>
<td>9</td>
<td>of order 16</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( H_l, 1 )</td>
<td>12</td>
<td>of order 8</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( Z_2, 1 )</td>
<td>3</td>
<td>of order 8</td>
</tr>
<tr>
<td>( Q, Z_1 )</td>
<td>( V_4 )</td>
<td>( S_3 )</td>
<td>( D_8, Z_2 )</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( G_1, 1 )</td>
<td>6</td>
<td>of order 8</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( G_2, 1 )</td>
<td>6</td>
<td>of order 8</td>
</tr>
</tbody>
</table>

**Case 2.** Let \( I = F_i \), and let \( J \in E \ \setminus \{ Q \} \). Then
<table>
<thead>
<tr>
<th>$I, J$</th>
<th>$I/J$</th>
<th>$Aut(I/J)$</th>
<th>$L, K$</th>
<th>No. of subgroups</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_i, F_i$</td>
<td>1</td>
<td>1</td>
<td>$D_8, D_8$</td>
<td>3 of order 32</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$G_k, G_k$</td>
<td>9 of order 16</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$H_t, H_t$</td>
<td>12 of order 8</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$Z_2, Z_2$</td>
<td>3 of order 8</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1, 1</td>
<td>3 of order 4</td>
</tr>
<tr>
<td>$F_i, Z_1$</td>
<td>$C_2$</td>
<td>1</td>
<td>$D_8, G_k$</td>
<td>9 of order 16</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$G_j, H_t$</td>
<td>12 of order 8</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$G_k, Z_2$</td>
<td>9 of order 8</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$H_t, 1$</td>
<td>12 of order 4</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$Z_2, 1$</td>
<td>3 of order 4</td>
</tr>
<tr>
<td>$F_i, 1$</td>
<td>$C_4$</td>
<td>$C_2$</td>
<td>$G_3, 1$</td>
<td>6 of order 4</td>
</tr>
</tbody>
</table>

**Case 3.** Let $I = Z_1$, and let $J \in E \setminus \{Q, F_i\}$. Then

<table>
<thead>
<tr>
<th>$I, J$</th>
<th>$I/J$</th>
<th>$Aut(I/J)$</th>
<th>$L, K$</th>
<th>No. of subgroups</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z_1, Z_1$</td>
<td>1</td>
<td>1</td>
<td>$D_8, D_8$</td>
<td>1 of order 16</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$G_k, G_k$</td>
<td>3 of order 8</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$H_t, H_t$</td>
<td>4 of order 4</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$Z_2, Z_2$</td>
<td>1 of order 4</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1, 1</td>
<td>1 of order 2</td>
</tr>
<tr>
<td>$Z_1, 1$</td>
<td>$C_2$</td>
<td>1</td>
<td>$D_8, G_k$</td>
<td>3 of order 8</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$G_j, H_t$</td>
<td>4 of order 4</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$G_k, Z_2$</td>
<td>3 of order 4</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$H_t, 1$</td>
<td>4 of order 2</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$Z_2, 1$</td>
<td>1 of order 2</td>
</tr>
</tbody>
</table>

**Case 4.** Let $I = 1 = J$. Then

<table>
<thead>
<tr>
<th>$I, J$</th>
<th>$I/J$</th>
<th>$Aut(I/J)$</th>
<th>$L, K$</th>
<th>No. of subgroups</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1, 1$</td>
<td>1</td>
<td>1</td>
<td>$D_8, D_8$</td>
<td>1 of order 8</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$G_k, G_k$</td>
<td>3 of order 4</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$H_t, H_t$</td>
<td>4 of order 2</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$Z_2, Z_2$</td>
<td>1 of order 2</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1, 1</td>
<td>1 of order 1</td>
</tr>
</tbody>
</table>
Hence, there are 189 subgroups of $Q \times D_8$, which include the trivial subgroup, 11 subgroups of order 2, 39 subgroups of order 4, 71 subgroups of order 8, 51 subgroups of order 16, 15 subgroups of order 32 and the group itself.

As in the construction of $Q \times Q$, one then uses this information to examine the groups of orders 2, 4, 8, 16 and 32 to construct its subgroup lattice. The details of this construction will not be included in this dissertation. However, the subgroup lattice of $Q \times D_8$ will be given in Section 3.3 for comparison with $Q \times Q$.

### 3.2 Extraspecial Groups

In this section, we define two types of groups, and we discuss how these groups are relevant to the subgroup lattices we have constructed. Two definitions from [2] are given here.

**Definition.** A group $G$ is called a central product of its subgroups $A$ and $B$ if

(i) $G = A \times B$

(ii) $[A, B] = 1$

This definition implies that $A, B \triangleleft G$ and that $A \cap B \leq Z(A) \cap Z(B)$. When $A$ and $B$ have isomorphic centers, we may also refer to a central product as a quotient of $G$ with amalgamated centers, and write it as $A \vartriangleleft B$.

**Definition.** Let $P$ be a $p$-group. We call $P$ extraspecial if $Z(P) = P' = \Phi(P)$.

There is a well known classification of extraspecial groups. In order to understand the classification of extraspecial groups which will follow in the next Theorem, the reader should be aware of the following notation and observations, given in [2].

For an odd prime $p$, let

$E = \langle x, y | x^p = y^p = 1, [x, y] \in Z(E) \rangle$, and

$F = \langle x, y | x^{p^2} = y^p = 1, [x, y] = x^p \rangle$.

These are the two extraspecial groups of order $p^3$.

When $p = 2$, the extraspecial groups of order $2^3$ are $D_8$ and $Q$. In the following Theorem, which classifies extraspecial groups, the notation and observations given above will be used.
Theorem 3.1. [2]

Let $p$ be a prime, and let $P$ be an extraspecial group of order $p^{2t+1}$. Then exactly one of the following four cases arises:

(i) $p \neq 2$, $\text{Exp}(P) = p$, and $P$ is a central product of $t$ copies of $E$;

(ii) $p \neq 2$, $\text{Exp}(P) = p^2$, and $P$ is a central product of $t - 1$ copies of $E$ with a copy of $F$;

(iii) $p = 2$, and $P$ is a central product of $t$ copies of $D_8$;

(iv) $p = 2$, and $P$ is a central product of $t - 1$ copies of $D_8$ with a copy of $Q$.

Using the above theorem, one is able to see that there are exactly two extraspecial groups of order 32. They are given by $Q \times Q$ and $Q \times D_8$. The subgroup lattices of these two extraspecial groups of order 32 lie in the subgroup lattices of $Q \times Q$ and $Q \times D_8$. The reader can recognize them as the quotient by the diagonal of the centers. In $Q \times Q$, the group that identifies the centers is $Z_2$, a cyclic group of order 2. Being able to apply Theorems 1.7 and 1.8 to construct these subgroup lattices of $Q \times Q$ and $Q \times D_8$ and to identify the extraspecial groups that lie within them was our goal. In the next section these subgroup lattices will be given.

3.3 Subgroup Lattices

In this section, the subgroup lattices of $Q \times Q$ and $Q \times D_8$ are given so one can compare them. Also, the subgroup lattices of the extraspecial groups of order 32 are given.
The subgroup lattice of $Q \times Q$ is:
The subgroup lattice of $Q \times D_8$ is:
The extraspecial group of order 32 from $Q \times Q$ is:

To determine the number of subgroups in $Q \gamma Q$, we use the table from Section 2.1 to count the subgroups that satisfy the following property: $Z_2$ is contained in a subgroup if and only if $Z \leq I$, $Z \leq L$ and $Z \leq J$ if and only if $Z \leq K$.

$Q \gamma Q$ has 110 subgroups, consisting of the following:

15 subgroups of order 16
35 subgroups of order 8
39 subgroups of order 4
19 subgroups of order 2, and
the trivial subgroup and the group itself.
The extraspecial group of order 32 from $Q \times D_8$ is:

To determine the number of subgroups in $Q \times D_8$, we use the table from Section 3.1 to count the subgroups that satisfy the following property: The group whose projections and intersections are given by $Z_1, 1$ and $Z_2, 1$ is contained in a subgroup if and only if $Z_1 \leq I$, $Z_2 \leq L$ and $Z_1 \leq J$ if and only if $Z_2 \leq K$.

$Q \times D_8$ has 78 subgroups, consisting of the following:

15 subgroups of order 16
35 subgroups of order 8
15 subgroups of order 4
11 subgroups of order 2, and
the trivial subgroup and the group itself.
Conclusions

We finish by discussing future projects and questions that our current work gave rise to. In the future, we would like to continue our work in characterizing subgroups of products of groups. The methods used to construct these lattices involved explicit details. Particularly, the details included repeated application of the results given in Section 1.3 to determine containment of subgroups from one level to the next in the subgroup lattices and knowledge of the subgroup structures of groups of orders 16 and 32. It would be interesting to know if this information can be generalized to provide more insight about subgroup lattices without doing such explicit work and also if the work already done might have deeper implications.

After investigating lattice properties of extraspecial groups in the case when \( p = 2 \), it would be interesting to explore lattice properties of extraspecial groups in the case when \( p > 2 \). When we started on this project of further characterizing subgroups in products of groups, we first studied and characterized subgroups of a central product. Our motivation for this dissertation was determined by our original question and goal, which was to determine when two central products were isomorphic. There are more generalized versions of a central product in [9] that make this question a plausible one. We are still working to answer this question. Studying the subgroup lattices of \( Q \times Q \), \( Q \times D_8 \) and the extraspecial groups of order 32 provided more insight and led to other questions that are worth pursuing. For example, what can one say about two nonisomorphic groups with the same subgroup lattice structure? Are the lattices of normal subgroups of the extraspecial groups of order 32 the same as their dual? I intend to revisit my original question and continue these characterizations. For comparison, with the information we have found, it would also be an interesting project to construct the subgroup lattice of \( D_8 \times D_8 \). Although we already know what the extraspecial group is going to be, it would be interesting to see if one could use the details from our constructions to learn more information about constructing subgroup lattices and to see if more information can be obtained from within that subgroup lattice.
In working with extraspecial groups, it was important to understand $p$-groups. Because $p$-groups play such a significant role in the study of group theory it would be interesting to study their structure.

Another interesting future project would be to see if a computer program could be written to construct these subgroup lattices because the processes used in our constructions were very methodical. Constructing the subgroup lattices was indeed tedious, as one can imagine, because of the number of subgroups involved. Although this work was tedious, seeing how the results could be applied to construct subgroup lattices that contain a significant amount of information and making this contribution to the area of finite group theory was gratifying. It was very rewarding to provide a property that further characterizes subgroups of a direct product and to find the subgroup lattices of the extraspecial groups of order 32. Providing them served as a bridge from our original question involving central products to the classification of extraspecial groups of order $p^{2t+1}$. 
Bibliography


