Non-triviality of Knots Arising from Iterated Infection Without the Use of the Tristram Levine Signature.

Chris Davis, Rice University

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Outline:

1. Background
   - The solvable filtration of the knot concordance group.
   - Infection.
   - Von Neumann $\rho$-invariants.

2. Statement of the theorem and an application.
   - Blanchfield form.

3. Proof
The solvable filtration of the knot concordance group.

Definition

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There is an operation called connected sum on knots. Under this operation the set of knots forms a commutative monoid.
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**Definition**

A knot sitting in $S^3 = \partial B^4$ is called slice if bounds an embedded locally flat disk in $B^4$.

Modulo slice knots, knots form an abelian group under connected sum, the Knot Concordance Group, $C$. 
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**Definition**

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Modulo slice knots, knots form an abelian group under connected sum, the Knot Concordance Group, $\mathcal{C}$.

Cochran-Orr-Teichner provided a filtration of $\mathcal{C}$ by subgroups:

$$\ldots \mathcal{F}_{1.5} \subseteq \mathcal{F}_1 \subseteq \mathcal{F}.5 \subseteq \mathcal{F}_0 \subseteq \mathcal{C}$$
The solvable filtration of the knot concordance group.

**Definition**

A knot is called \((n)\)-solvable, denoted \(K \in \mathcal{F}_n\), if there exists a smooth 4-manifold \(W\), called a \((n)\)-solution, bounded by \(M(K)\) such that:

1. The inclusion induced map \(H_1(M(K)) \to H_1(W)\) is an isomorphism.
2. \(H_2(W)\) has a basis given by smoothly embedded surfaces \(L_i\) and \(D_i\) all disjoint except that \(L_i \cap D_i = \text{one point}\).
3. \(\pi_1(L_i)\) sits in the \(n\)'th term in the derived series of \(\pi_1(W)\), \(\pi_1(W)^{(n)}\).
4. \(\pi_1(D_i)\) sits in the \(n\)'th term in the derived series of \(\pi_1(W)\).

the derived series of a group \(G\) is defined by \(G^{(0)} = G\) and \(G^{(n+1)} = [G^{(n)}, G^{(n)}]\).
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A knot is called \((n.5)\)-solvable, denoted \(K \in \mathcal{F}_{n.5}\), if there exists a smooth 4-manifold \(W\), called a \((n.5)\)-solution, bounded by \(M(K)\) such that:

1. The inclusion induced map \(H_1(M(K)) \rightarrow H_1(W)\) is an isomorphism.
2. \(H_2(W)\) has a basis given by smoothly embedded surfaces \(L_i\) and \(D_i\) all disjoint except that \(L_i \cap D_i = \) one point
3. \(\pi_1(L_i)\) sits in the \(n + 1\)'th term in the derived series of \(\pi_1(W)\).
4. \(\pi_1(D_i)\) sits in the \(n\)'th term in the derived series of \(\pi_1(W)\).

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Infection

Take knots $R$ and $J$ and an unknotted curve $\eta$ in $E(R)$ (the exterior of $R$). Infection of $R$ by $J$ along $\eta$ ($R_{\eta}(J)$) can be described as taking the strands of $R$ which pass through the disk bounded by $\eta$ and tying them into the knot $J$. 
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\[\begin{align*}
\text{+1} & & \text{+2} & & \text{-1} & & \eta \\
R & & & & & & \\
J & & & & & & \\
\end{align*}\]
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\[ R_{\eta}(J) \]
Infection

When $R$ is slice and $\eta$ has zero linking with $R$, the pair $(R, \eta)$ is sometimes called a “doubling operator.” This condition is assumed throughout this talk. Such infection increases solvability:

**Theorem (Cochran-Orr-Teichner)**

*If $R, \eta$ is a doubling operator, and $J$ is $(n)$-solvable, then $R\eta(J)$ is $(n + 1)$-solvable.*

Iterating the infection process yields that if $J$ is $(n)$-solvable, then $(R\eta)^k(J)$ is $(n + k)$-solvable.
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I will call the knot $J$ the deepest infecting knot.
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- Cochran-Orr-Teichner and Cochran-Teichner perform a single infection along curves deep in the derived series by knots for which the integral of the Tristram-Levine signature of $J$ is greater than the Cheeger-Gromov invariant of $R$. The resulting knots are shown to be on infinite order in $\mathcal{F}_n/\mathcal{F}_{n.5}$

- Cochran-Harvey-Leidy constructed knots of the form $(R_{\eta})_n(J)$ which are of linearly independent in $\mathcal{F}_n/\mathcal{F}_{n.5}$ under the condition that the integral of the Tristram-Levine signature of $J$ is rationally independent of the first order signatures of $R$.

- Cha constructed knots of the form $(R_{\eta})_n(J)$ which are linearly independent in $\mathcal{F}_n/\mathcal{F}_{n.5}$ under the condition that the Tristram-Levine signature of $J$ evaluated at $p$'th roots of unity is greater than the Cheeger-Gromov invariant of $R$. 
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- Cha constructed knots of the form $(R_\eta)^n(J)$ which are linearly independent in $F_n/F_{n.5}$ under the condition that the Tristram-Levine signature of $J$ evaluated at $p'$th roots of unity is greater than the Cheeger-Gromov invariant of $R$. 

Von Neumann $\rho$ Invariants.

Given connected closed oriented 3-manifolds $M_1 \ldots M_n$ and coefficient systems $\phi_i : \pi_1(M_i) \to \Gamma_i$, suppose there exists a compact oriented 4-manifold $W$ with $\partial W = \sqcup M_i$ and a coefficient system $\psi : \pi_1(W) \to \Lambda$ such that there are monomorphisms $\alpha_i : \Gamma_i \to \Lambda$ making following diagram commute

$$
\begin{array}{ccc}
\pi_1(M_i) & \xrightarrow{\phi_i} & \Gamma_i \\
\downarrow i_* & & \downarrow \alpha_i \\
\pi_1(W) & \xrightarrow{\psi} & \Lambda 
\end{array}
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$$

Definition

The von Neumann $\rho$ invariant is defined by

$$
\sum_{i=1}^{n} \rho(M_i, \phi_i) = \sigma^{(2)}(W, \psi) - \sigma(W)
$$
How are $\rho$-invariants, infection and solvability related?
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- some $\rho$-invariants vanish on knots in $\mathcal{F}_{n.5}$:

**Proposition (Cochran-Orr-Teichner)**

If $J$ is $(n.5)$-solvable and $\phi : \pi_1(M(J)) \to \Gamma$ is a PTFA (Poly-Torsion-Free-Abelian) coefficient system with $\Gamma^{(n)} = 0$ which extends over an $(n.5)$-solution $W$ then $\rho(M(J); \phi) = \sigma^2(W, \phi) - \sigma(W) = 0$. 

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- $\rho$-invariants add under infection:

**Proposition**

$-M(R), -M(J)$ and $M(R_\eta(J))$ cobound a 4-manifold $V$. If $\phi$ is a PTFA coefficient system on $V$, then

\[
\rho(M(R_\eta(J)), \phi) - \rho(M(R), \phi) - \rho(M(J), \phi) = \sigma^{(2)}(V, \phi) - \sigma(V) = 0.
\]

Thus, $\rho(M(R_\eta(J)), \phi) = \rho(M(R), \phi) + \rho(M(J), \phi)$. 
Examples: “Abelian” $\rho$-invariants

For a knot $J$, $\rho(M(J), \phi)$ is given by:

1. the integral of the Tristram-Levine signature if $\phi: \pi_1(M(J)) \to \mathbb{Z}$ is the abelianization map.
2. the sum of the Tristram-Levine signature evaluated at the $p$'th roots of 1 if $\phi: \pi_1(M(J)) \to \mathbb{Z} \to \mathbb{Z}/p$ is the abelianization map followed by mod $p$ reduction.

The first is used in the infection results of Cochran-Orr-Teichner and Cochran-Harvey-Leidy. The second is used in the infection results of Cha.
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The second is used in the infection results of Cha.
Invariants of Interest: $\rho^1$

For $K$ a knot, let $\phi^1 : \pi_1(M(J)) \rightarrow \frac{\pi_1(M(J))}{\pi_1(M(J))(2)}$ be the quotient map.
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**Definition**

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- Has a polynomial splitting property similar to the first order $\rho$-invariants of S. Kim-T. Kim.
- Not a concordance invariant. One of the goals of this talk is to get concordance information out of it anyway.
Then for any $n$, $(R_\eta)^n(J)$ represents an element of infinite order in $\mathcal{F}_{n-.5}/\mathcal{F}_{n+1.5}$. Where $\mathcal{F}_{-.5}$ is taken to be all of $\mathbb{C}$. 
Statement of main theorem

Theorem (D.)

Let $R$ be a slice knot.

Then for any $n$, $(R_\eta)^n(J)$ represents an element of infinite order in $\mathcal{F}_{n-1.5}/\mathcal{F}_{n+1.5}$. Where $\mathcal{F}_{-1.5}$ is taken to be all of $\mathbb{C}$. 

Linear Independence.

Why $\mathcal{F}_{n-1.5}/\mathcal{F}_{n+1.5}$ instead of $\mathcal{F}_n/\mathcal{F}_{n+1.5}$? $n+1.5$: $\rho_1$ is an obstruction to torsion modulo $1.5$-solvability, so for $n=0$ the theorem should obstruct $1.5$-solvability.

$n-1.5$: A knot with prime Alexander polynomial is not $(1.5)$-solvable. In the examples I will present, the deepest infecting knot $J$ is not even $(0)$-solvable.
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Let $R$ be a slice knot. Let $\eta$ be curve in $E(R)$ which represents a “doubly-anisotropic” element of the rational Alexander module of $R$.

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Linear Independence.

Why $F_{n-.5}/F_{n+1.5}$ instead of $F_{n}/F_{n+1}$.?

$\rho^1$ is an obstruction to torsion modulo $1/2$-solvability, so for $n=0$ the theorem should obstruct $1/2$-solvability. In the examples I will present, the deepest infecting knot $J$ is not $(0)$-solvable.
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  - \( n - .5 \): A knot with prime Alexander polynomial is not (.5)-solvable. In the examples I will present, the deepest infecting knot \( J \) is not even (0)-solvable.
Consider the slice knot $R$ with doubly anisotropic curve $\eta$. 
Application

Consider the slice knot $R$ with doubly anisotropic curve $\eta$. 

$+1$  

$+2$  

$-1$  

$R$  

$\eta$
Consider the slice knot $R$ with doubly anisotropic curve $\eta$. The $-7$ twist knot has non-zero $\rho^1$-invariant (D.) but vanishing Tristram-Levine signature.
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The condition on $\eta$: Doubly Anisotropic

**Definition (D.)**

An Element of the Alexander module of a knot $A_0(K)$, is called doubly anisotropic if it does not sit in the sum of any pair of isotropic submodules with respect to the classical Blanchfield linking form.
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In the previous example, $R$ has Alexander module $A_0(R) = \mathbb{Z}[t, t^{-1}]/(\delta(t)^2)$ where $\delta(t) = t^2 - 3t + 1$ is a prime.
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$\eta$ is a generator, so it is doubly anisotropic.

This condition is neither stronger nor weaker than the robust doubling operators of Cochran-Harvey-Leidy (though the example on the previous page is both.)
The classical Blanchfield Form

For a knot $K$, there is a nonsingular Hermitian form

$$BL : A_0(K) \times A_0(K) \rightarrow \mathbb{Q}(t)/\mathbb{Z}[t, t^{-1}].$$
The classical Blanchfield Form

For a knot $K$, there is a nonsingular Hermitian form

$$BL : A_0(K) \times A_0(K) \to \mathbb{Q}(t)/\mathbb{Z}[t, t^{-1}].$$

If a knot is slice (or even (0.5)-solvable) and $E$ is a slice disk compliment (or (0.5)-solution) for $K$, then $P = \ker(A_0(K) \to A_0(E))$ is isotropic. That is, for all $a, b \in P$, $BL(a, b) = 0.$
Doubly anisotropic:
Why? A proof of a baby version of the theorem

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\[
\begin{array}{c}
A \\
V \\
B \\
M(J)
\end{array}
\]
A proof of a baby version of the theorem

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Inclusion induces a map from \( A_0(R) = H_1(M(R); \mathbb{Z}[t, t^{-1}]) \) to \( H_1(W; \mathbb{Z}[t, t^{-1}]) \).
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Its kernel is given by the sum of the kernels of the maps \( A_0(R) \rightarrow H_1(A; \mathbb{Z}[t, t^{-1}]) \) and \( A_0(R_\eta(J)) \rightarrow H_1(B; \mathbb{Z}[t, t^{-1}]) \). Notice that these are both isotropic.
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Thus \( \eta \) does not sit in this kernel and \( \mu_J = \eta \) is nonzero in \( H_1(W; \mathbb{Z}[t, t^{-1}]) \). Thus the following diagram commutes

\[
\begin{array}{ccc}
\pi_1(M(J)) & \longrightarrow & \langle \mu \rangle = \mathbb{Z} \\
\downarrow & & \uparrow \\
\pi_1(W) & \longrightarrow & \Lambda = \mathbb{Z} \times H_1(W; \mathbb{Z}[t, t^{-1}])
\end{array}
\]
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\begin{array}{ccc}
\pi_1(M(J)) & \xrightarrow{\phi^1} & \mathbb{Z} \ltimes A_0(J) \\
\downarrow & & \downarrow \\
\pi_1(W) & \longrightarrow & \Lambda' := \Lambda \ltimes H_1(W; \mathbb{Z}[\Lambda])
\end{array}
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Applying Novikov Additivity and that \( \Lambda' \) is PTFA we see that

\[ \rho(M(J), \mathbb{Z} \times A_0(J)) = \sigma^{(2)}(A, \Lambda') + \sigma^{(2)}(B, \Lambda') + \sigma^{(2)}(V, \Lambda') \]
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Moreover, since \( \Lambda'(3) = 1 \) A could be take to be a \((2.5)\)-solution and this would still hold.
The baby version of the Theorem:

Thus,

**Proposition**

*If* $R$ *is a slice knot,* $\eta$ *is an unknotted curve in* $E(R)$ *doubly anisotropic and* $K$ *is a knot with prime Alexander polynomial and nonzero* $\rho^1$-*invariant, then* $R_\eta(K)$ *is nontrivial in* $\mathcal{F}_{5}/\mathcal{F}_{2.5}$
The proof of the full version of the Theorem

Start with the 4-manifold from the previous proof and then keep on adding cobordisms to it.
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Suppose that \((R_\eta)^n(K) = R_\eta((R_\eta)^{n-1}(K))\) is \((n + 1.5)\)-solvable. Let \(A\) be an \((n + 1.5)\)-solution for it. Let \(B\) be a slice disk compliment for \(R\).
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- Similarly cap the boundary component. The double anisotropy condition will guarantee that the meridian of
  \((R_\eta)^{n-2}(K)\)
  is nontrivial in \(H_1(W; \mathbb{Z}[\Lambda])\). Use this to build another PTFA coefficient system, \(\Lambda'\) into which the meridian injects.
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- repeat.
closing remarks

This only shows that this knot is not \((n + 1)\)-solvable. In order to see that no multiple of it is, a similar cobordism is used. The \(\rho_1\)-obstruction enjoys a splitting property which can be used to get a linear independence claim.
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