Localized metabelian rho invariants as obstructions to torsion in the knot concordance group.

Chris Davis, Rice University

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Goals

1. Introduce some $\rho$-invariants corresponding to localizations of the Alexander module of a knot.
2. Prove that they obstruct knots which are torsion in the algebraic concordance group from being torsion in the concordance group.
3. Prove computable bounds on the value of these invariants.
4. Do a computation to prove the theorem about twist knots which is on the next slide.
An Application: Twist Knots

**Theorem (D.)**

No nontrivial linear combination of the twist knots, $T_7, T_{13}, T_{21} \ldots T_{x^2+x+1}$ $(x \geq 2)$, is topologically slice. These knots are each of algebraic order 2.

Figure: $T_n$: the $n$-twist knot.
Previous Twist Knot Results

**Theorem (A. Tamulis, 2000)**

No nontrivial combination of the twist knots $T_n$ with $n \geq 3$ and $4n + 1$ prime is topologically slice. Each of these twist knots is algebraically of order 2.

**Theorem (C. Livingston, 2001)**

Let $p_i$ be an enumeration of the primes congruent to 3 mod 4. Let $n_i = p_{2i-1}p_{2i} - 1$. No nontrivial linear combination of the knots $T_{n_i}$ is slice. Each of these twist knots is algebraically of order 4.

**Theorem (S. Kim, 2005)**

No nontrivial linear combination of the twist knots, except for the 0 twist knot (unknot), the 1 twist knot (figure eight) and the 2 twist knot (stevedore) is ribbon.
Preliminaries: von Neumann $\rho$ Invariants.

Given connected closed oriented 3-manifolds $M_1 \ldots M_n$ and coefficient system $\phi_i : \pi_1(M_i) \to \Gamma_i$, suppose there exists a compact oriented 4-manifold $W$ with $\partial W = \sqcup M_i$ and a coefficient system $\psi : \pi_1(W) \to \Lambda$ such that there are monomorphisms $\alpha_1 : \Gamma_i \to \Lambda$ making following diagram commute.

\[
\begin{array}{ccc}
\pi_1(M_i) & \xrightarrow{\phi_i} & \Gamma_i \\
\downarrow i_* & & \downarrow \alpha_i \\
\pi_1(W) & \xrightarrow{\psi} & \Lambda
\end{array}
\]
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\end{array}
\]

Definition

The von Neumann $\rho$ invariant is defined by

\[
\sum_{i=1}^n \rho(M_i, \phi_i) = \sigma^{(2)}(W, \psi) - \sigma(W).
\]
Preliminaries: Two Facts About $\sigma^{(2)}$

When $\mathbb{Z}[\Lambda]$ is an Ore domain, $|\sigma^{(2)}(W,\psi)| \leq \text{rank} \mathbb{Z}[\Lambda](H^2(W;\mathbb{Z}[\Lambda])^*)[H^2(\partial W;\mathbb{Z}[\Lambda])]$.

If $\Lambda = \mathbb{Z}$ and $A$ is a matrix with polynomial entries representing the twisted intersection form of $W$, then $\sigma^{(2)}(W,\psi) = \int S_1 \ldots \int S_1 \sigma(A(\omega)) \, d\omega$ where $A(\omega)$ is the hermitian matrix given by evaluating the entries of $A$ at $\omega$, $\sigma(A(\omega))$ is its classical (integer valued) signature and the integral is taken with respect to normalized Lebesgue measure.
Preliminaries: Two Facts About $\sigma^{(2)}$

- When $\mathbb{Z}[\Lambda]$ is an Ore domain,
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- If $\Lambda = \mathbb{Z}^n$ and $A$ is a matrix with polynomial entries representing the twisted intersection form of $W$, then
  $$\sigma^{(2)}(W; \psi) = \int_{S^1} \ldots \int_{S^1} \sigma(A(\omega)) d\omega$$

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Invariants of Interest: $\rho^0$

For $L$ an $n$ component link with zero linking numbers, $M(L)$ denotes zero surgery along $L$. 

Let $\phi: \pi_1(M(L)) \to \pi_1(M(L))$ be the abelianization map. 

Definition $\rho^0(L) := \rho(M(L), \phi)$

For a knot it is the integral of the Tristram-Levine Signature. (Cochran-Orr-Teichner '04) For links it is computable. (More on this later)
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For $K$ a knot, let $\phi^1 : \pi_1(M(K)) \to \frac{\pi_1(M(K))}{\pi_1(M(K))^{(2)}}$ be the quotient map.
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- Similar coefficient systems were considered by Cochran-Harvey-Leidy ’08. If a knot is slice, at least one of these coefficient systems will extend.
- Hard to compute. There is no known procedure.
Invariants of Interest: $\rho^1_p$

Let $p(t)$ be a polynomial. $R_p := \{ f, g \in \mathbb{Q}(t) : (g, p) = 1 \}$ is the localization of $\mathbb{Q}[t]_{\pm 1}$ at $p$.

Let $A_{p0}(K) = A_0(K) \otimes R_p$ be the localized Alexander module of $K$.

Let $\pi_1(M(K))_p(t)$ be the kernel of the composition $\pi_1(M(K))_1 \to \pi_1(M(K))_1 \to A_{p0}(K) \to \text{Id} \otimes 1 A_{p0}(K)$.

Let $\phi_1: \pi_1(M(K)) \to \pi_1(M(K)) \to A_{p0}(K)$ be the quotient map.
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Let $A_0^p(K) = A_0(K) \otimes R_p$ be the localized Alexander module of $K$.

Let $\pi_1(M(K))_{p(t)}^{(2)}$ be the kernel of the composition

$$\pi_1(M(K))^{(1)} \to \frac{\pi_1(M(K))^{(1)}}{\pi_1(M(K))^{(2)}} \hookrightarrow A_0(K) \to A_0^p(K).$$
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Let $A^p_0(K) = A_0(K) \otimes R_p$ be the localized Alexander module of $K$.

Let $\pi_1(M(K))^{(2)}_{p(t)}$ be the kernel of the composition

$$\pi_1(M(K))^{(1)} \to \frac{\pi_1(M(K))^{(1)}}{\pi_1(M(K))^{(2)}} \hookrightarrow A_0(K) \to A^p_0(K).$$

Let $\phi^1_p : \pi_1(M(K)) \to \frac{\pi_1(M(K))}{\pi_1(M(K))^{(2)}_{p}}$ be the quotient map.
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Let $\phi^1_p : \pi_1(M(K)) \rightarrow \frac{\pi_1(M(K))}{\pi_1(M(K))^{(2)}_{p}}$ be the quotient map.

**Definition**

$$\rho^1_{p(t)}(K) := \rho(M(K), \phi^1_p)$$
A Pair of Examples. 1: Localize Away from the Alexander Polynomial

Let $\Delta$ be the Alexander polynomial of $K$.

Suppose $(p, \Delta) = 1$. Then $\frac{1}{\Delta} \in R_p$. $\Delta$ annihilates $A_0$, so $A_0 \otimes R_p = A_0^p$ vanishes.

$\pi_1(M(K))_{p(t)}^{(2)}$ is the kernel of the composition

$$\pi_1(M(K))^{(1)} \to \pi_1(M(K))^{(1)}_{\pi_1(M(K))^{(2)}} \hookrightarrow A_0(K) \to A_0^p(K) = 0.$$

$\pi_1(M(K))_{p(t)}^{(2)} = \pi_1(M(K))^{(1)}$, so $\rho_1^p(K) = \rho^0(K)$. 
A Pair of Examples. 2: Localize at the Alexander Polynomial

Suppose $p = \Delta$. Then $A_0(K) \xrightarrow{\text{id} \otimes 1} A_0(K) \otimes R_p$ is injective.

$\pi_1(M(K))^{(2)}_{p(t)}$ is the kernel of the composition

$$
\pi_1(M(K))^{(1)} \xrightarrow{\pi_1(M(K))^{(1)}} \frac{\pi_1(M(K))^{(1)}}{\pi_1(M(K))^{(2)}} \xhookrightarrow{} A_0(K) \xhookrightarrow{} A_0^p(K).
$$

$\pi_1(M(K))^{(2)}_{p(t)} = \pi_1(M(K))^{(2)}$, so $\rho^1_p(K) = \rho^1(K)$. 
Proposition (D.)

Let $\Delta$ be the Alexander polynomial of a knot, $K$.

- For any $p$ which is relatively prime to $\Delta$, $\rho_p^1(K) = \rho^0(K)$.
- $\rho_\Delta^1(K) = \rho^1(K)$. 

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\( \rho_P^1 \) Under Connected Sum and Infection

**Proposition (D.)**

\[ \rho_P^1(J \# K) = \rho_P^1(J) + \rho_P^1(K). \]

**Proposition (D.)**

For \( \eta \) an unknot representing an element of the localized Alexander module of \( J \), \( \rho_P^1(J_\eta(K)) = \begin{cases} 
\rho_P^1(J) & \text{if } \eta = 0 \text{ in } A_0^P(J) \\
\rho_P^1(J) + \rho^0(K) & \text{if } \eta \neq 0 \text{ in } A_0^P(J) 
\end{cases} \)
Under Connected Sum and Infection

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\end{cases}$

From the second proposition I get the following discouraging example:
A Pair of Slice Knots with Differing $\rho^1$ Invariants

Consider the pair of slice knots below. $\rho^1(R_\eta(A)) = \rho^1(R) + \rho^0(A)$. In particular $\rho^1(R)$ and $\rho^1(R_\eta(A))$ cannot both be zero. $\rho^1$ is not well defined on the concordance group.

Figure: A pair of slice knots with differing $\rho^1$. 
A Pair of Slice Knots with Differing $\rho^1$ Invariants

Consider the pair of slice knots below. $\rho^1(R_\eta(A)) = \rho^1(R) + \rho^0(A)$. In particular $\rho^1(R)$ and $\rho^1(R_\eta(A))$ cannot both be zero. $\rho^1$ is not well defined on the concordance group. Despite this fact, I will provide in the next few slides a setting in which these invariants provide useful concordance information.

![Figure: A pair of slice knots with differing $\rho^1$.](image-url)
The Blanchfield Form and Isotropy

\( \mathbb{Q}[t^{\pm 1}] \), and thus \( R_p \) when \( p \) is symmetric, have an involution given by

\[ \overline{q}(t) = q(t^{-1}) \]
The Blanchfield Form and Isotropy

$\mathbb{Q}[t^{\pm 1}]$, and thus $R_p$ when $p$ is symmetric, have an involution given by

$q(t) = q(t^{-1})$

With respect to this involution, the localized Alexander module has a hermitian form

$$Bl^p : A^p_0(K) \times A^p_0(K) \rightarrow \frac{\mathbb{Q}(t)}{R_p}.$$ 

given by extending the Blanchfield form on $A_0(K)$.
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**Definition**

$P \subseteq A_0^p(K)$ is called **isotropic** if $Bl_p(a, b) = 0$ for all $a, b \in P$
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**Definition**

\(K\) is called **\(p\)-anisotropic** if there is no nontrivial isotropic submodule of \(A_0^p(K)\).
Exploration of Anisotropy.

If $p$ is a symmetric prime which divides $\Delta$ with multiplicity 1,
Exploration of Anisotropy.

If $p$ is a symmetric prime which divides $\Delta$ with multiplicity 1, then $A_0^p(K)$ is cyclic: $A_0^p(K) = \frac{R_p}{(p)}$. 

Let $\alpha$ generate $A_0^p(K)$. The Blanchfield form is given by $\text{Bl}_p(q\alpha, r\alpha) = qra$ with $(a, p) = 1$. If $\text{Bl}_p(q\alpha, r\alpha) = 0$ then $p$ divides $q$ or $r$. By symmetry of $p$, $p | q$ if and only if $p | q$. Thus $\text{Bl}_p(a, b) = 0$ if and only if $a$ or $b$ is zero in $A_0^p(K)$. In particular there are no nontrivial isotropic submodules. This proves the following in the case that $p$ is prime.

Proposition

If the Alexander polynomial of a knot is squarefree, then the knot is $p$-anisotropic for every symmetric polynomial, $p$. 

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This proves the following in the case that $p$ is prime.

**Proposition**

*If the Alexander polynomial of a knot is squarefree, then the knot is $p$-anisotropic for every symmetric polynomial, $p$.***
First Big Theorem: $\rho^1_p$ as an Obstruction to Linear Dependence in $\mathcal{C}$

Theorem (D.)

Given a symmetric polynomial $p(t)$ and (not necessarily distinct) $p$-anisotropic knots $K_1, \ldots, K_n$, if $K_1 \# \cdots \# K_n$ is slice, then

$$\sum_{i=1}^{n} \rho^1_{p(t)}(K_i) = 0$$
First Big Theorem: $\rho_p^1$ as an Obstruction to Linear Dependence in $\mathcal{C}$

**Theorem (D.)**

*Given a symmetric polynomial $p(t)$ and (not necessarily distinct) $p$-anisotropic knots $K_1, \ldots, K_n$, if $K_1 \# \ldots \# K_n$ is slice, then*

$$\sum_{i=1}^n \rho_p^1(p(t)(K_i)) = 0$$

**Corollary (D.)**

*If $K$ is of finite concordance order and is $p$-anisotropic, then $\rho_p^1(K) = 0$*
First Big Theorem: $\rho^1_p$ as an Obstruction to Linear Dependence in $C$

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If $K_1, \ldots, K_n$ are of finite algebraic order and have coprime, squarefree Alexander polynomials and if $\rho^1(K_i)$ is nonzero for each $i$, then the set \{ $K_1, \ldots, K_n$ \} is linearly independent in $C$. 
Proof of the Second Corollary

Let $p_j$ be the Alexander polynomial of $K_j$. 

Since these knots have coprime Alexander polynomials, $\rho_1 p_j(K_i) = \rho_0(K_i)$ if $j \neq i$ and $\rho_1 p_j(K_j) \neq 0$. Since each of the knots are of finite algebraic order, $\rho_0(K_i) = 0$. Thus, $0 = a_j \rho_1(K_j)$. Since $\rho_1(K_j) \neq 0$, $a_j = 0$.

No nontrivial combinations of the knots $K_i$ is slice and the set is linearly independent.
Proof of the Second Corollary

Let \( p_j \) the Alexander polynomial of \( K_j \). If \( \#a_iK_i \) is slice, then

\[
0 = \sum_{i=1}^{n} a_i \rho_{p_j}^{1}(K_i) \quad \text{for each } j.
\]
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Proof of the Second Corollary

Let $p_j$ the Alexander polynomial of $K_j$. If $\#a_iK_i$ is slice, then

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$$0 = a_j \rho^1(K_j).$$

Since $\rho^1(K_j) \neq 0$, $a_j = 0$. No nontrivial combinations of the knots $K_i$ is slice and the set is linearly independent.
Consider a family of knots of finite topological order, $J_1, J_2, \ldots$ with distinct, prime Alexander polynomials.

$\rho_1(J_n) = 0$

Let $K_n$ be given from $J_n$ by infection along a curve which is nonzero in $A_0(J_n)$ using a knot $A$ with $\rho_0(A) \neq 0$.

$\rho_1(K_n) = \rho_1(J_n) + \rho_0(A) = \rho_0(A) \neq 0$

by the second corollary, \{ $K_1, K_2, \ldots$ \} is linearly independent.
Example: A Linearly Independent Family of Knots in $C$.

- Consider a family of knots of finite topological order, $J_1, J_2, \ldots$ with distinct, prime Alexander polynomials.
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\[ J_n \]
Example: A Linearly Independent Family of Knots in \( \mathcal{C} \).

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- by the second corollary, $\{K_1, K_2, \ldots\}$ is linearly independent.
Proof of First Big Theorem

I start by constructing a 4-manifold whose signature defect is the sum of $\rho$-invariants in question.

Take $M(K_1) \times [0,1] \sqcup ... \sqcup M(K_n) \times [0,1]$ and connect it by gluing together neighborhoods of meridians.

Glue a slice disk compliment for $K_1 \# ... \# K_n$ to the $M(K_1 \# ... \# K_n)$-component of the boundary.

It is the anisotropic assumption that lets us cap the cobordism safely.

$M(K_1) M(K_2) M(K_3) M(K_1 \# K_2 \# K_3) W_0 W V$

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Proof of First Big Theorem

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Proof of First Big Theorem

A coefficient system on $\mathcal{W}$ must be found.

\[ \pi_1(\mathcal{W}) \rightarrow \pi_1(M(K_i)) \rightarrow \pi_1(M(K_i))^{(2)} \]
Proof of First Big Theorem

Every inclusion on first homology is an isomorphism. (So $A_0^p(W) = H_1(W; R_p)$ makes sense.)

\[
\begin{align*}
\pi_1(V) & \longrightarrow H_1(V) = \mathbb{Z} \\
\uparrow & \quad \uparrow \cong \\
\pi_1(M(\#K_i)) & \rightarrow H_1(M(\#K_1)) = \mathbb{Z} \\
\downarrow & \quad \downarrow \cong \\
\pi_1(W_0) & \longrightarrow H_1(W_0) = \mathbb{Z} \\
\uparrow & \quad \uparrow \cong \\
\pi_1(M(K_i)) & \rightarrow H_1(M_i) = \mathbb{Z}
\end{align*}
\]
Claim: $\ker(A^0_0(K_i) \to A^0_0(W))$ is isotropic and thus, zero.
Proof of First Big Theorem

Claim: \( \ker(A_0^p(K_i) \to A_0^p(W)) \) is isotropic and thus, zero.

\[ P = \ker \left( (A_0^p(\#K_i) = \oplus A_0^p(K_i)) \to A_0^p(V) \right) \] is isotropic.  \((V\) is a slice disk complement.)
Proof of First Big Theorem

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\( P = \ker((A_0^p(\#K_i) = \bigoplus A_0^p(K_i)) \to A_0^p(V)) \) is isotropic. \((V \text{ is a slice disk complement.})\)

\[
\begin{align*}
A_0^p(W) &\xrightarrow{\sim} \bigoplus A_0^p(K_j) \\
A_0^p(W_0) &\xrightarrow{\sim} \bigoplus A_0^p(K_j) \\
A_0^p(K_i) &\xrightarrow{\sim} [0,...,0,1,0...]
\end{align*}
\]
Proof of First Big Theorem

\[
\frac{\pi_1(M(K))}{\pi_1(M(K))^{(2)}_p} = H_1(M(K_i)) \ltimes A_0^p(K_i)\mathbb{Z}, \text{ so the isomorphism and monomorphism on the previous two slides give the desired monomorphism.}
\]

\[
\begin{array}{ccc}
\pi_1(W) & \longrightarrow & \frac{\pi_1(W)}{\pi_1(W)^{(2)}_p} \\
\uparrow & & \uparrow \\
\pi_1(M(K_i)) & \rightarrow & \frac{\pi_1(M(K_i))}{\pi_1(M(K_i))^{(2)}_p}
\end{array}
\]
Thus, \( \sum \rho_p^1(K_i) = \sigma^2(W, \phi_p^1) - \sigma(W) \).
\( \sigma^{(2)}(W; \phi_p^1) = \sigma(W) = 0 \) since it turns out that the twisted and untwisted second homology of \( W \) is carried by its boundary.
Twist Knots

Need to compute $\rho^1(T_{x^2+x+1})$.
Twist Knots

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Twist Knots

Need to compute $\rho^1(T_{x^2+x+1})$. This is hard. The infection trick won’t work. Instead try to compute $2\rho^1(T_n) = \rho^1(T_n \# T_n)$. This is algebraically slice and its Seifert form has a metabolizing link. The following theorem will allow us to get a handle on $\rho^1$
Second Big Theorem: Computing $\rho^1$.

Theorem ($D.$)

Let $K_1, \ldots, K_n$ be (not necessarily distinct) $p$-anisotropic knots. Suppose $\# K_i$ is algebraically slice. Then

$$\left| \sum \rho^1_p(K_i) - \rho^0(L) \right| \leq g - 1 - \eta(L).$$

$\rho^1_p$ is an approximated by $\rho^0$, which is more reasonable to compute.
Second Big Theorem: Computing \( \rho^1 \).

**Theorem (D.)**

Let \( K_1, \ldots, K_n \) be (not necessarily distinct) \( p \)-anisotropic knots. Suppose \( \# K_i \) is algebraically slice. Let \( F \) be a genus \( g \) seifert surface for \( \# K_i \). Let \( F \) be a genus \( g \) seifert surface for \( \# K_i \). Let \( \eta(L) \) be the rank of the Alexander module of \( L \).

Then

\[
|\sum_{i=1}^n \rho^1_p(K_i) - \rho^0(L)| \leq g - 1 - \eta(L).
\]

\( \rho^1 \) (a nonabelian \( \rho \)-invariant) is approximated by \( \rho^0 \) (an abelian \( \rho \)-invariant). The latter is more reasonable to expect to compute.
Theorem (D.)

Let $K_1, \ldots, K_n$ be (not necessarily distinct) $p$-anisotropic knots. Suppose $n \nabla K_i$ is algebraically slice. Let $F$ be a genus $g$ Seifert surface for $\nabla K_i$. Let $L$ be a link of $g$ curves on $F$ representing a metabolizer for the Seifert form,

$$\rho^1_p((\nabla K_i)) - \rho^0(L) \leq g - 1 - \eta(L).$$

$\rho^1_p$ (a nonabelian $\rho$-invariant) is approximated by $\rho^0$ (an abelian $\rho$-invariant). The latter is more reasonable to expect to compute.
Second Big Theorem: Computing $\rho^1$.

**Theorem (D.)**

Let $K_1, \ldots, K_n$ be (not necessarily distinct) $p$-anisotropic knots. Suppose $\# K_i$ is algebraically slice. Let $F$ be a genus $g$ Seifert surface for $\# K_i$. Let $L$ be a link of $g$ curves on $F$ representing a metabolizer for the Seifert form, such that the meridians about the bands on which the components of $L$ sit form a $\mathbb{Z}$-linearly independent set in $A_{0}^p(\#K_i)/P$, where $P$ is the submodule of $A_{0}^p(\#K_i)$ generated by $L$.

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$$\left| \sum_{i=1}^n \rho_1^p(K_i) - \rho_0^p(L) \right| \leq g - 1 - \eta(L).$$

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Then

$$\left| \left( \sum \rho^1_p(K_i) \right) - \rho^0(L) \right| \leq g - 1 - \eta(L).$$
Second Big Theorem: Computing $\rho^1$.

**Theorem (D.)**

Let $K_1, \ldots, K_n$ be (not necessarily distinct) $p$-anisotropic knots. Suppose $\# K_i$ is algebraically slice. Let $F$ be a genus $g$ seifert surface for $\# K_i$. Let $L$ be a link of $g$ curves on $F$ representing a metabolizer for the Seifert form, such that the meridians about the bands on which the components of $L$ sit form a $\mathbb{Z}$-linearly independent set in $A^p_0(\# K_i)/P$, where $P$ is the submodule of $A^p_0(\# K_i)$ generated by $L$. Let $\eta(L)$ be the rank of the Alexander module of $L$. Then

$$\left| \left( \sum \rho^1_p(K_i) \right) - \rho^0(L) \right| \leq g - 1 - \eta(L).$$

$\rho^1_p$ (a nonabelian $\rho$-invariant) is approximated by $\rho^0$ (an abelian $\rho$-invariant). The latter is more reasonable to expect to compute.
More Examples: Infection by a String Link.

Let $L$ be a 2 component link with zero linking numbers. $J_n(L) \# J_n$ is algebraically slice.

\[
|\rho_1(J_n(L)) + \rho_1(J_n) - \rho_0(L)| \leq 2 - 1 - \eta(L) \leq 1.
\]

Thus, if $|\rho_0(L)| > 1$, then $\rho_1(J_n(L)) + \rho_1(J_n) \neq 0$.

As in the previous example, $\rho_1(J_n) = 0$.

Thus, $\rho_1(J_n(L)) \neq 0$ and $\{J_n(L) : n \geq 1\}$ is linearly independent.
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Let $L$ be a 2 component link with zero linking numbers. $J_n(L) \# J_n$ is algebraically slice. It has $L$ as a metabolizer.
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\[ |\rho^1(J_n(L)) + \rho^1(J_n) - \rho^0(L)| \leq 2 - 1 - \eta(L) \leq 1. \]
More Examples: Infection by a String Link.

Let $L$ be a 2 component link with zero linking numbers. $J_n(L) \neq J_n$ is algebraically slice. It has $L$ as a metabolizer.

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\[
\begin{array}{c}
\begin{array}{c}
\text{\textbf{\textit{J}}}_n(\text{\textbf{\textit{L}}}) \\
/ \backslash \\
\text{n} / \text{-n}
\end{array}
\end{array}
\begin{array}{c}
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Localized metabelian rho invariants as obstructions to torsion in the knot concordance group.  
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More Examples: Infection by a String Link.

Let $L$ be a 2 component link with zero linking numbers. $J_n(L) \not\cong J_n$ is algebraically slice. It has $L$ as a metabolizer.

$$\left| \rho^1(J_n(L)) + \rho^1(J_n) - \rho^0(L) \right| \leq 2 - 1 - \eta(L) \leq 1.$$ Thus, if $|\rho^0(L)| > 1$, then $\rho^1(J_n(L)) + \rho^1(J_n) \neq 0$. As in the previous example, $\rho^1(J_n) = 0$. Thus, $\rho^1(J_n(L)) \neq 0$ and $\{J_n(L) : n \geq 1\}$ is linearly independent.
The 4-manifold Used to Prove the Second Big Theorem

Start with $W_0$ as in the previous proof, consider $L$ as a link in $\partial^+ W_0$. Add 2-handles to the zero framing of $L$. Adding a three handle to this manifold results in a manifold with boundary $(\sqcup M(K_i)) \sqcup -M(L)$. $M(K_1) \sqcup M(K_2) \sqcup M(K_3) \sqcup (L)$. 

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The 4-manifold Used to Prove the Second Big Theorem

Start with $W_0$ as in the previous proof,

\[ M(K_1 \# K_2 \# K_3) \]
The 4-manifold Used to Prove the Second Big Theorem

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$M(K_1 \# K_2 \# K_3)$

$M(K_1)$ $M(K_2)$ $M(K_3)$
The 4-manifold Used to Prove the Second Big Theorem

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![Diagram](image-url)
Start with $W_0$ as in the previous proof, consider $L$ as a link in $\partial_+ W_0$. Add 2-handles to the zero framing of $L$. Adding a three handle to this manifold results in a manifold with boundary $(\sqcup M(K_i)) \sqcup -M(L)$
The 4-manifold used to prove the second big theorem

\[
\begin{align*}
\pi_1(M(L)) & \longrightarrow H_1(M(L)) \\
\pi_1(W) & \longrightarrow \frac{\pi_1(W)}{\pi_1(W)^{(2)}} \\
\pi_1(M(K_i)) & \longrightarrow \frac{\pi_1(M(K_i))}{\pi_1(M(K_i))^{(2)}}
\end{align*}
\]
The 4-manifold used to prove the second big theorem

Thus,
\[ \sum \rho_p^1(K_i) - \rho^0(L) = \sigma^2(W, \phi_p^1) - \sigma(W) \]
The 4-manifold used to prove the second big theorem

Thus,
\[ \sum \rho^1_p(K_i) - \rho^0(L) = \sigma^2(W, \phi^1_p) - \sigma(W) \]

\[ \sigma(W) = 0 \]

\[ |\sigma^2(W, \phi^1_p)| \leq \text{rank} \left( \frac{H_2(W; \Gamma)}{H_2(\partial W; \Gamma)} \right) \]

\[ = g - 1 - \eta(L) \]

where \( \Gamma = \mathbb{Q} \left[ \frac{\pi_1(W)}{\pi_1(W)^{(2)}_P} \right] \)
Proof of Main Application.

For $n = x^2 + x + 1$, $T_n \# T_n$ is algebraically slice.
Proof of Main Application.

For $n = x^2 + x + 1$, $T_n \# T_n$ is algebraically slice. It has a metabolizer which I call $L_x$. (Such a nice metabolizer is why I consider only these twist knots.)
Proof of Main Application.

For $n = x^2 + x + 1$, $T_n \# T_n$ is algebraically slice. It has a metabolizer which I call $L_x$. (Such a nice metabolizer is why I consider only these twist knots.) Thus we are in the setting of the second big theorem.

\[ x \text{ red strands} \quad x + 1 \text{ blue strands} \]
Computation.

- If $|\rho^0(L_x)| > 1$ then $\rho^1(T_n) \neq 0$, since $|2\rho^1(T_n) - \rho^0(L_x)| < 2 - 1 - 0$ and the proof of linear independence will be complete, since they have distinct, prime Alexander polynomials.
Computation.

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- The computation of $\rho^0(L_x)$ can be reduced to planar considerations, algorithmized, and implemented on a computer.
Computation.

- If $|\rho^0(L_x)| > 1$ then $\rho^1(T_n) \neq 0$, since $|2\rho^1(T_n) - \rho^0(L_x)| < 2 - 1 - 0$ and the proof of linear independence will be complete, since they have distinct, prime Alexander polynomials.
- The computation of $\rho^0(L_x)$ can be reduced to planar considerations, algorithmized, and implemented on a computer.
- I will do the computations by hand for an easier example, and then state the results for $L_x$. In the end, $\rho^0(L_x) < -1$ for each $x$, and the proof will be complete.
Simplified Example

Let $K$ be the knot with the surgery description below. It happens to be the left handed trefoil.
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- Let $K$ be the knot with the surgery description below. It happens to be the left handed trefoil.
- This surgery description gives $M(K)$ as the boundary of the 4-manifold, $W$. The associated inclusion is an isomorphism on first homology.
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- This surgery description gives $M(K)$ as the boundary of the 4-manifold, $W$. The associated inclusion is an isomorphism on first homology.
- Thus, $\rho^0(K) = \sigma^2(W, \phi) - \sigma(W)$.

\[ \begin{array}{c}
\includegraphics[width=0.5\textwidth]{simplified_example_diagram}
\end{array} \]
Simplified Example

- Let $K$ be the knot with the surgery description below. It happens to be the left handed trefoil.
- This surgery description gives $M(K)$ as the boundary of the 4-manifold, $W$. The associated inclusion is an isomorphism on first homology.
- Thus, $\rho^0(K) = \sigma^2(W, \phi) - \sigma(W)$.
- By inspection, $\sigma(W) = -1$. 

\[ \text{\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{knot_diagram.png}
\caption{Left handed trefoil knot diagram}
\end{figure}} \]
Computing $\sigma^2(W)$:

The curve, $\gamma$, lifts to the universal abelian cover of $M(\text{unknot})$ where it bounds an embedded disk, $D$. This disk together with the core of the 2-handle glued to $\gamma$ form the 2-sphere $S$ which generates $H_2(W; \mathbb{Q}[\mathbb{Z}])$. The twisted intersection form of $W$ is given by the self intersection of $S$. The self intersection of $S$ in $H_2(W; \mathbb{Q}[\mathbb{Z}])$ is the same as the intersection (with coefficients) between $D$ and the $-1$ push-off of $\gamma$ in the cover of $M(\text{unknot})$. Counting these, $\langle S, S \rangle = -t - t^{-1} + 1$, so that $[-t - t^{-1} + 1]$ is the twisted intersection matrix of $W$. 

$-1$ $\gamma$
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![Diagram showing the curve $\gamma$ lifted to the universal abelian cover, forming a 2-sphere $S$ with a 2-handle, and the twisted intersection form of $W$ calculuated.]
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\[ \langle S, S \rangle = -t - t^{-1} + 1 \]
Computing $\sigma^2(W)$:

Let $f(t) = 1 - t - t^{-1}$. Now using the second fact about $L^2$-signatures,

$$\sigma^2(W) = \int_{S^1} \sigma([f(\omega)])d\omega = \int_{S^1} |1 - 2 \text{Re}(\omega)| d\omega = \frac{1}{3}.$$

Thus, $\rho^0(K) = \frac{1}{3} - (-1) = \frac{4}{3}$.
Computation for the Link, $L_2$

In general this can be duplicated for any link arising as $\pm 1$ surgery along commutator curves on the unlink.
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$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
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and twisted intersection form (computed by taking the algorithm outlined for the trefoil knot and implementing it on a computer)

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\begin{pmatrix}
1 + xy^2 + y + x^{-1}y^{-2} + y^{-1} & -xy - y^{-2} & xy + y^{-2} \\
-y^2 - x^{-1}y^{-1} & y + y^{-1} & -y + x^{-1} - y^{-1} \\
y^2 + x^{-1}y^{-1} & -y + x - y^{-1} & y + y^{-1}
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which have signatures 4 and $\sim .38$ (numerically integrating via computer).
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$$

which have signatures 4 and $\sim .38$ (numerically integrating via computer). Thus, $\rho^0(L_2) \sim -3.62$ and $2\rho^1(T_7) \lesssim -2.62$ ($7 = 2^2 + 2 + 1$), and in particular is not zero.
Computation for the Link, $L_x$

$L_{x+1}$ can be realized from $L_x$ by $+1$ surgery along $2x - 1$ new commutator curves.
Computation for the Link, \( L_x \)

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**Proposition (D.)**

*If a link \( L' \) is realized as +1 surgery along commutator curves on another link \( L \), then \( \rho^0(L') \leq \rho^0(L) \).

The knots have distinct, prime Alexander polynomials and so are linearly independent.
Computation for the Link, $L_x$

$L_{x+1}$ can be realized from $L_x$ by +1 surgery along $2x - 1$ new commutator curves.

**Proposition (D.)**

If a link $L'$ is realized as +1 surgery along commutator curves on another link $L$, then $\rho^0(L') \leq \rho^0(L)$

Thus $\rho^0(L_x) \leq \rho^0(L_2) \lesssim -3.62$ for all $x \geq 2$ and $\rho^1(T_{x^2+x+1}) \neq 0.$
Computation for the Link, $L_x$

$L_{x+1}$ can be realized from $L_x$ by +1 surgery along $2x - 1$ new commutator curves.

**Proposition (D.)**

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Thus $\rho^0(L_x) \leq \rho^0(L_2) \lesssim -3.62$ for all $x \geq 2$ and $\rho^1(T_{x^2+x+1}) \neq 0$. The knots have distinct, prime Alexander polynomials and so are linearly independent.
Thank you.