Localized metabelian rho invariants as obstructions to torsion in the knot concordance group.

Chris Davis, Rice University

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Goals

1. Introduce some $\rho$-invariants corresponding to localizations of the Alexander module of a knot.
2. Prove that they obstruct knots which are torsion in the algebraic concordance group from being torsion in the concordance group.
3. Prove computable bounds on the value of these invariants.
4. Do a computation to prove the theorem about twist knots which is on the next slide.
5. State a theorem regarding these invariants and iterated infection.
An application: Twist Knots

**Theorem (D.)**

No nontrivial linear combination of the twist knots, $T_7, T_{13}, T_{21} \ldots T_{x^2+x-+} \ (x \geq 2)$, is topologically slice. These knots are each of algebraic order 2.

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![Diagram of $T_n$: the $n$-twist knot.](image_url)
Previous related results

**Theorem (A. Tamulis, 2000)**

No nontrivial combination of the twist knots $T_n$ with $n \geq 3$ and $4n + 1$ prime is topologically slice. Each of these twist knots is algebraically of order 2.

**Theorem (C. Livingston, 2001)**

Let $p_i$ be an enumeration of the primes congruent to 3 mod 4. Let $n_i = p_{2i-1}p_{2i} - 1$. No nontrivial linear combination of the knots $T_{n_i}$ is slice. Each of these twist knots is algebraically of order 4.

**Theorem (S. Kim, 2005)**

No nontrivial linear combination of the twist knots, except for the 0 twist knot (unknot), the 1 twist knot (figure eight) and the 2 twist knot (stevedore) is ribbon.
Theorem

For a sufficiently nice slice knot, $R$, and a sufficiently nice infecting curve, $\eta$, the set resulting from iterated infection of $R$ along $\eta$ by each of $T_7, T_{21}, \ldots, T_{x^2+x+1}$ is linearly independent in the concordance group.
Twist knots and infection

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Example of niceness: any slice knot with Alexander module of the form $\frac{\mathbb{Q}[t^{\pm 1}]}{p(t)^2}$ with $p(t)$ prime and symmetric and $\eta$ a generator of the Alexander module.

$\rho^0(T_{x^2+x+1}) = 0$
Invariants of interest: $\rho^0$

For $L$ an $n$ component link with zero linking numbers, $M(L)$ denotes zero surgery along $L$. 

Let $\phi: \pi_1(M(L)) \to \pi_1(M(L))$ be the Abelianization map.

Definition $\rho^0(L) := \rho(M(L), \phi)$

For a knot it is the integral of the Tristram-Levine Signature. (Cochran-Orr-Teichner '04)

For links it is computable: In a nice enough case a computer can be taught how to do it. (More on this later)
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For $K$ a knot, let $\phi^1 : \pi_1(M(K)) \to \frac{\pi_1(M(K))}{\pi_1(M(K))^{(2)}}$ be the quotient map.
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$$\rho^1(K) := \rho(M(K), \phi^1)$$

- Considered by Cochran-Harvey-Leidy ’08 together with quotients corresponding to isotropy of the Blanchfield form as a slice obstruction.
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- Considered by Cochran-Harvey-Leidy '08 together with quotients corresponding to isotropy of the Blanchfield form as a slice obstruction.
- Hard to compute. There is no known procedure.
Invariants of interest: $\rho_p^1$
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$$R_p := \left\{ \frac{f}{g} \in \mathbb{Q}(t) : (g, p) = 1 \right\}$$

is the localization of $\mathbb{Q}[t^{\pm 1}]$ at $p$. 
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Let $A_0^p(K) = A_0(K) \otimes R_p$ be the localized Alexander module of $K$. 

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Let $\pi_1(M(K))^{(2)}_{p(t)}$ be the kernel of the composition

$$\pi_1(M(K))^{(1)} \rightarrow \frac{\pi_1(M(K))^{(1)}}{\pi_1(M(K))^{(2)}} \hookrightarrow A_0(K) \overset{\text{Id} \otimes 1}{\longrightarrow} A_0^p(K).$$
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Let $\phi^1_p : \pi_1(M(K)) \rightarrow \frac{\pi_1(M(K))}{\pi_1(M(K))_{p(t)^{(2)}}}$ be the quotient map.

**Definition**

$$\rho^1_{p(t)}(K) := \rho(M(K), \phi^1_p)$$
A pair of examples. 1: localize away from the Alexander polynomial

Let $\Delta$ be the Alexander polynomial of $K$.

Suppose $(p, \Delta) = 1$. Then $\frac{1}{\Delta} \in R_p$. $\Delta$ annihilates $A_0$, so $A_0 \otimes R_p = A_0^p$ vanishes.

$\pi_1(M(K))_{p(t)}^{(2)}$ is the kernel of the composition

$$\pi_1(M(K))^{(1)} \to \frac{\pi_1(M(K))^{(1)}}{\pi_1(M(K))^{(2)}} \hookrightarrow A_0(K) \to A_0^p(K) = 0.$$ 

$\pi_1(M(K))_{p(t)}^{(2)} = \pi_1(M(K))^{(1)}$, so $\rho_1^p(K) = \rho_0^0(K)$.
A pair of examples. 2: localize at the Alexander polynomial

Suppose $p = \Delta$. Then $A_0(K) \to A_0(K) \otimes R_p$ is injective.

$\pi_1(M(K))^{(2)}_{p(t)}$ is the kernel of the composition

$$
\pi_1(M(K))^{(1)} \to \frac{\pi_1(M(K))^{(1)}}{\pi_1(M(K))^{(2)}} \hookrightarrow A_0(K) \hookrightarrow A_0^p(K).
$$

$\pi_1(M(K))^{(2)}_{p(t)} = \pi_1(M(K))^{(2)}$, so $\rho_1^p(K) = \rho_1^1(K)$
Interactions between $\rho_p^1$ and the Alexander polynomial

Proposition (D.)

Let $\Delta$ be the Alexander polynomial of a knot, $K$.

- For any $p$ which is relatively prime to $\Delta$, $\rho_p^1(K) = \rho^0(K)$.
- $\rho_\Delta^1(K) = \rho^1(K)$. 

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\[ \rho_p^1(J \# K) = \rho_p^1(J) + \rho_p^1(K). \]

**Proposition (D.)**

For \( \eta \) an unknot representing an element of the localized Alexander module of \( J \),

\[ \rho_p^1(J_\eta(K)) = \begin{cases} 
\rho_p^1(J) & \text{if } \eta = 0 \text{ in } A_0^p(J) \\
\rho_p^1(J) + \rho^0(K) & \text{if } \eta \neq 0 \text{ in } A_0^p(J) 
\end{cases} \]
\( \rho_p^1 \) under infection and connected sums

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From the second proposition I get the following discouraging example:
A pair of slice knots with differing $\rho^1$ invariants

Consider the pair of slice knots below. $\rho^1(R_\eta(A)) = \rho^1(R) + \rho^0(A)$. In particular $\rho^1(R)$ and $\rho^1(R_\eta(A))$ cannot both be zero. $\rho^1$ is not well defined on the concordance group.

![Diagram of slice knots](image)

**Figure:** A pair of slice knots with differing $\rho^1$. 
A pair of slice knots with differing $\rho^1$ invariants

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Despite this fact, I will provide in the next few slides a setting in which these invariants provide useful concordance information.

Figure: A pair of slice knots with differing $\rho^1$. 
Blanchfield forms and isotropy

$\mathbb{Q}[t^{\pm 1}]$, and thus $R_p$ when $p$ is symmetric, have an involution given by

$$\overline{q}(t) = q(t^{-1})$$
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With respect to this involution, the localized Alexander module has a hermitian form

\[
Bl^p : A_p^0(K) \times A_p^0(K) \to \frac{\mathbb{Q}(t)}{R_p}.
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given by extending the Blanchfield form on \( A_0(K) \)
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**Definition**

\( P \subseteq A_0^p(K) \) is called isotropic if \( BL^p(a, b) = 0 \) for all \( a, b \in P \).
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**Definition**

\( K \) is called **\( p \)-anisotropic** if there is no nontrivial isotropic submodule of \( A^p_0(K) \).
Exploration of anisotropy.

If $p$ is a symmetric prime which divides $\Delta$ with multiplicity 1,
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If $p$ is a symmetric prime which divides divides $\Delta$ with multiplicity 1, then $A_p^0(K)$ is cyclic: $A_p^0(K) = \frac{R_p}{(p)}$. 

Let $\alpha$ generate $A_p^0(K)$. The Blanchfield form is given by $Bl_p(q \alpha, r \alpha) = qra^p$ with $(a, p) = 1$. If $Bl_p(q \alpha, r \alpha) = 0$ then $p$ divides $q$ or $r$. By symmetry of $p$, $p | q$ if and only if $p | q$. Thus $Bl_p(a, b) = 0$ if and only in $a$ or $b$ is zero in $A_p^0(K)$. In particular there are no nontrivial isotropic submodules. This proves the following in the case that $p$ is prime.

Proposition If the Alexander polynomial of a knot is squarefree, then the knot is $p$-anisotropic for every symmetric polynomial, $p$. 

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The Blanchfield form is given by $Bl^p(q\alpha, r\alpha) = \frac{\overline{q}ra}{p}$ with $(a, p) = 1$. If $Bl^p(q\alpha, r\alpha) = 0$ then $p$ divides $\overline{q}$ or $r$. By symmetry of $p$, $p|\overline{q}$ if and only if $p|q$.
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This proves the following in the case that \( p \) is prime.

**Proposition**

*If the Alexander polynomial of a knot is squarefree, then the knot is \( p \)-anisotropic for every symmetric polynomial, \( p \).*
First big theorem: $\rho_p^1$ as an obstruction to linear dependence in $C$

**Theorem (D.)**

Given a symmetric polynomial $p(t)$ and (not necessarily distinct) $p$-anisotropic knots $K_1, \ldots, K_n$, if $K_1 \# \ldots \# K_n$ is slice, then

$$\sum_{i=1}^n \rho_{p(t)}^1(K_i) = 0$$
First big theorem: \( \rho_p^1 \) as an obstruction to linear dependence in \( \mathcal{C} \)

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**Corollary (D.)**

*If \( K \) is of finite concordance order and is \( p \)-anisotropic, then \( \rho_p^1(K) = 0 \)
First big theorem: $\rho^1_p$ as an obstruction to linear dependence in $\mathcal{C}$

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If $K_1, \ldots, K_n$ are of finite algebraic order and have coprime, squarefree Alexander polynomials, then if $\rho^1(K_i)$ is nonzero for each $i$, $\{K_1, \ldots, K_n\}$ is linearly independent in $\mathcal{C}$. 
Proof of second corollary

Let $p_j$ the Alexander polynomial of $K_j$. 
Proof of second corollary

Let $p_j$ the the Alexander polynomial of $K_j$. If $\# a_i K_i$ is slice, then

$$0 = \sum_{i=1}^{n} a_i \rho^1_{p_j}(K_i)$$

for each $j$. Since these knots have coprime Alexander polynomials, $\rho^1_{p_j}(K_i) = \rho^0(K_i)$ if $j \neq i$ and $\rho^1_{p_j}(K_j) \neq 0$. Since each of the knots are of finite algebraic order, $\rho^0(K_i) = 0$. Thus, $0 = a_j \rho^1_{p_j}(K_j)$. Since $\rho^1_{p_j}(K_j) \neq 0$, $a_j = 0$. No nontrivial combinations of the knots $K_i$ is slice and the set is linearly independent.
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Since $\rho^1(K_j) \neq 0$, $a_j = 0$.

No nontrivial combinations of the knots $K_i$ is slice and the set is linearly independent.
Comparison of second corollary to polynomial splitting theorems

The second corollary can be compared to the polynomial splitting theorems for Casson-Gordon invariants (S. Kim, 2003) and for $\rho$-invariants corresponding to isotropy in the Alexander module (S. Kim-T. Kim, 2007). The idea of the comparison: Each theorem says that if some knots with coprime Alexander polynomials have non-vanishing obstructions then their connected sum has non-vanishing obstructions.
Example: a linearly independent family of knots in $\mathcal{C}$ via infection.

- Consider a family of knots of finite topological order, $J_1, J_2, \ldots$ with distinct, prime Alexander polynomials.
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\[ A \quad n \quad -n \]

$K_n$
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by the second corollary, $\{K_1, K_2, \ldots \}$ is linearly independent.
Proof of first big theorem

I start by constructing a 4-manifold whose signature defect is the sum of $\rho$-invariants in question. Take $M(K_1) \times [0,1] \sqcup ... \sqcup M(K_n) \times [0,1]$ and connect it by gluing together neighborhoods of meridians. Glue a slice disk compliment for $K_1 \# ... \# K_n$ to the $M(K_1 \# ... \# K_n)$-component of the boundary. It is the anisotropic assumption that lets us cap the cobordism safely.

$M(K_1)M(K_2)M(K_3)M(K_1 \# K_2 \# K_3)W_0W_V$ coral reef upside-down = jellyfish
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\[
\begin{array}{ccc}
W & V \\
\downarrow & \downarrow \\
W_0 & \\
\downarrow & \downarrow \\
M(K_1) & M(K_2) & M(K_3)
\end{array}
\]

coral reef upside-down = jellyfish
Proof of first big theorem

A coefficient system on $W$ must be found.

\[ \pi_1(W) \rightarrow \pi_1(M(K_i)) \rightarrow \pi_1(M(K_i))^{(2)} \]
Proof of first big theorem

Every inclusion on first homology is an isomorphism. (So $A_0^p(W) = H_1(W; R_p)$ makes sense.)

\[ \pi_1(V) \xrightarrow{\sim} H_1(V) = \mathbb{Z} \]
\[ \pi_1(M(#K_i)) \xrightarrow{\sim} H_1(M(#K_1)) = \mathbb{Z} \]
\[ \pi_1(W_0) \xrightarrow{\sim} H_1(W_0) = \mathbb{Z} \]
\[ \pi_1(M(K_i)) \xrightarrow{\sim} H_1(M_i) = \mathbb{Z} \]
Proof of first big theorem

Claim: \( \ker(A^p_0(K_i) \rightarrow A^p_0(W)) \) is isotropic and thus, zero.
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Claim: \( \ker(A_0^p(K_i) \to A_0^p(W)) \) is isotropic and thus, zero.
\[ P = \ker(A_0^p(\#K_i) \to A_0^p(V)) \] is isotropic.
Proof of first big theorem

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\( P = \ker (A_0^p(\#K_i) \to A_0^p(V)) \) is isotropic.

\[
\begin{align*}
A_0^p(V) & \xrightarrow{\cong} \bigoplus A_0^p(K_j) \\
\bigoplus A_0^p(K_j) & \xrightarrow{\cong} A_0^p(\#K_i) \\
A_0^p(\#K_i) & \xrightarrow{\cong} A_0^p(W_0) \\
A_0^p(W_0) & \xrightarrow{\cong} A_0^p(K_i)
\end{align*}
\]
Proof of first big theorem

\[
\frac{\pi_1(K)}{\pi_1(K)_{(2)}^p} = H_1(M(K_i)) \ltimes A_0^p(K_i) \mathbb{Z}
\]
Proof of first big theorem

Thus, \[ \sum \rho_p^1(K_i) = \sigma^2(W, \phi_p^1) - \sigma(W) = 0 \]
Twist knots

Need to compute $\rho^1(T_{x^2+x+1})$. 
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This is hard. The infection trick won’t work. Instead try to compute
$2\rho^1(T_n) = \rho^1(T_n \# T_n)$. 

\begin{tikzpicture}
\draw (0,0) circle (1);
\draw (1,0) circle (1);
\draw (2,0) circle (1);
\draw (3,0) circle (1);
\draw (4,0) circle (1);
\draw (5,0) circle (1);
\draw (6,0) circle (1);
\draw (7,0) circle (1);
\draw (8,0) circle (1);
\draw (9,0) circle (1);
\draw (0,1) circle (1);
\draw (1,1) circle (1);
\draw (2,1) circle (1);
\draw (3,1) circle (1);
\draw (4,1) circle (1);
\draw (5,1) circle (1);
\draw (6,1) circle (1);
\draw (7,1) circle (1);
\draw (8,1) circle (1);
\draw (9,1) circle (1);
\node at (0.5,0.5) {$-n$};
\node at (2.5,0.5) {$+1$};
\node at (5.5,0.5) {$-n$};
\node at (7.5,0.5) {$+1$};
\end{tikzpicture}
Twist knots

Need to compute $\rho^1(T_{x^2+x+1})$. This is hard. The infection trick won’t work. Instead try to compute $2\rho^1(T_n) = \rho^1(T_n\# T_n)$. This is algebraically slice, and has a metabolizing link. The following theorem will allow us to get a handle on $\rho^1$.
Second big theorem: computing $\rho^1$.

**Theorem (D.)**

Let $K_1, \ldots, K_n$ be (not necessarily distinct) $p$-anisotropic knots. Suppose $\# K_i$ is algebraically slice.

Let $F$ be a genus $g$ Seifert surface for $\# K_i$. Let $L$ be a link of $g$ curves on $F$ representing a metabolizer for the Seifert form, such that the meridians about the bands on which the components of $L$ sit form a $\mathbb{Z}$-linearly independent set in $A_{p}^0(\# K_i)/P$, where $P$ is the submodule of $A_{p}^0(\# K_i)$ generated by $L$.

Let $\eta(L)$ be the rank of the Alexander module of $L$. Then

$$\left| \sum_{i=1}^{n} \rho^1_p(K_i) - \rho^0(L) \right| \leq g - 1 - \eta(L).$$

$\rho^1_p$ (a nonabelian $\rho$-invariant) is approximated by $\rho^0$ (an abelian $\rho$-invariant). The latter is more reasonable to expect to compute.
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Then $\left| \sum (\rho^1(K_i)) - \rho^0(L) \right| \leq g - 1 - \eta(L)$. $\rho^1$ (a nonabelian $\rho$-invariant) is approximated by $\rho^0$ (an abelian $\rho$-invariant). The latter is more reasonable to expect to compute.
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$$\left|\sum_{i=1}^n \rho^1_{K_i} - \rho^0(L)\right| \leq g - 1 - \eta(L).$$

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More examples: Infection by a string link.

Let $L$ be a 2 component link with zero linking numbers. $J_n(L) \# J_n$ is algebraically slice.
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Let $L$ be a 2 component link with zero linking numbers. $J_n(L) \# J_n$ is algebraically slice. It has $L$ as a metabolizer.

$$|\rho^1(J_n(L)) + \rho^1(J_n) - \rho^0(L)| \leq 2 - 1 - \eta(L) \leq 1.$$
More examples: Infection by a string link.

Let $L$ be a 2 component link with zero linking numbers. $J_n(L) \# J_n$ is algebraically slice. It has $L$ as a metabolizer.

\[ |\rho^1(J_n(L)) + \rho^1(J_n) - \rho^0(L)| \leq 2 - 1 - \eta(L) \leq 1. \]

Thus, if $|\rho^0(L)| > 1$, then $\rho^1(J_n(L)) + \rho^1(J_n) \neq 0$. 

\[ \rho^1(J_n) = 0. \]

Thus, $\rho^1(J_n(L)) \neq 0$ and \{ $J_n(L)$ : $n \geq 1$ \} is linearly independent.
More examples: Infection by a string link.

Let \( L \) be a 2 component link with zero linking numbers. \( J_n(L) \neq J_n \) is algebraically slice. It has \( L \) as a metabolizer.

\[
|\rho^1(J_n(L)) + \rho^1(J_n) - \rho^0(L)| \leq 2 - 1 - \eta(L) \leq 1.
\]

Thus, if \( |\rho^0(L)| > 1 \), then \( \rho^1(J_n(L)) + \rho^1(J_n) \neq 0 \). As in the previous example, \( \rho^1(J_n) = 0 \).
More examples: Infection by a string link.

Let $L$ be a 2 component link with zero linking numbers. $J_n(L) \# J_n$ is algebraically slice. It has $L$ as a metabolizer.

$$|\rho^1(J_n(L)) + \rho^1(J_n) - \rho^0(L)| \leq 2 - 1 - \eta(L) \leq 1.$$ Thus, if $|\rho^0(L)| > 1$, then $\rho^1(J_n(L)) + \rho^1(J_n) \neq 0$. As in the previous example, $\rho^1(J_n) = 0$. Thus, $\rho^1(J_n(L)) \neq 0$ and $\{J_n(L) : n \geq 1\}$ is linearly independent.
Comparison to Casson-Gordon invariants and Metabelian $\rho$ invariants corresponding to isotropy

- **(advantage)** $\rho_p^1$ is not defined in terms of metabolizers/isotropic submodules: only one computation needs to be made, instead of (potentially infinitely) many.

- **(disadvantage)** Only applies to knots whose every prime factor is $p$-anisotropic. In particular, this technique cannot say anything about prime algebraically slice knots, for which the other invariants work very well.
Start with $W_0$ as in the previous proof, consider $L$ as a link in $\partial^+ W_0$. Add 2-handles to the zero framing of $L$. Adding a three handle to this manifold results in a manifold with boundary $(\sqcup M(K_i)) \sqcup -M(L)$.

\[ M(K_1) \sqcup M(K_2) \sqcup M(K_3) \sqcup (K_1 \# K_2 \# K_3) \sqcup L \]
The 4-manifold used to prove the second big theorem

Start with $W_0$ as in the previous proof,

$$M(K_1 \# K_2 \# K_3)$$
The 4-manifold used to prove the second big theorem

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Start with $W_0$ as in the previous proof, consider $L$ as a link in $\partial_+ W_0$. Add 2-handles to the zero framing of $L$.

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![Diagram](image-url)
The 4-manifold used to prove the second big theorem

\[ \pi_1(M(L)) \rightarrow H_1(M(L)) \]
\[ \pi_1(W) \rightarrow \frac{\pi_1(W)}{\pi_1(W)^{(2)}} \]
\[ \pi_1(M(K_i)) \rightarrow \frac{\pi_1(M(K_i))}{\pi_1(M(K_i))^{(2)}} \]
The 4-manifold used to prove the second big theorem

Thus,
\[ \sum \rho_p^1(K_i) - \rho^0(L) = \sigma^2(W, \phi_p^1) - \sigma(W) \]
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Thus,
\[ \sum \rho_p^1(K_i) - \rho^0(L) = \sigma^2(W, \phi_p^1) - \sigma(W) \]

\[ \sigma(W) = 0 \]
\[ |\sigma^2(W, \phi_p^1)| \leq \text{rank} \left( \frac{H_2(W; \Gamma)}{H_2(\partial W; \Gamma)} \right) \]
\[ = g - 1 - \eta(L) \]

where \( \Gamma = \mathbb{Q} \left[ \frac{\pi_1(W)}{\pi_1(W)^{(2)}_p} \right] \)
Proof of main application.

For \( n = -x^2 - x - 1 \), \( T_n \# T_n \) is algebraically slice.
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Proof of main application.

For \( n = -x^2 - x - 1 \), \( T_n \# T_n \) is algebraically slice. It has a metabolizer which I call \( L_x \). Thus we are in the setting of the second big theorem.
Computation.

- If $|\rho^0(L_x)| > 1$ then $\rho^1(T_n) \neq 0$, since $|2\rho^1(T_n) - \rho^0(L_x)| < 2 - 1 - 0$ and the proof of linear independence will be complete, since they have distinct, prime Alexander polynomials.
Computation.

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- The computation of $\rho^0(L_x)$ can be reduced to planar considerations, algorithmized, and implemented on a computer.

- I will do the computations by hand for an easier example, and then state the results for $L_x$. In the end, $\rho^0(L_x) < -1$ for each $x$, and the proof will be complete.
Simplified example

- Let $K$ be the knot with the surgery description below. It happens to be the left handed trefoil.
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- This surgery description gives $M(K)$ as the boundary of the 4-manifold, $W$. The associated inclusion is an isomorphism on first homology.
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- This surgery description gives $M(K)$ as the boundary of the 4-manifold, $W$. The associated inclusion is an isomorphism on first homology.
- Thus, $\rho^0(K) = \sigma^2(W, \phi) - \sigma(W)$.

\[ \begin{diagram}
  \node{\text{\textbullet}}
  \arrow{东南, blue}{-1}
  \node{\text{\textbullet}}
  \end{diagram} \]
Simplified example

- Let \( K \) be the knot with the surgery description below. It happens to be the left handed trefoil.
- This surgery description gives \( M(K) \) as the boundary of the 4-manifold, \( W \). The associated inclusion is an isomorphism on first homology.
- Thus, \( \rho^0(K) = \sigma^2(W, \phi) - \sigma(W) \).
- By inspection, \( \sigma(W) = -1 \).

\[
\begin{align*}
-1
\end{align*}
\]
Computing $\sigma^2(W)$:
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- the curve, $\gamma$, lifts to the universal abelian cover of $M(\text{unknot})$
Computing $\sigma^2(W)$:

- the curve, $\gamma$, lifts to the universal abelian cover of $M(\text{unknot})$ where it bounds an embedded disk, $D$. This disk together with the core of the 2-handle glued to $\gamma$ form the 2-sphere $S$ which generates $H_2(W; \mathbb{Q}[\mathbb{Z}])$. 

\[ \langle S, S \rangle = -t - t^{-1} + 1 \]

This is the twisted intersection matrix of $W - \gamma$. 

\[ \left[ -t - t^{-1} + 1 \right] \]
Computing $\sigma^2(W)$:

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Computing $\sigma^2(W)$:

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- The self intersection of $S$ in $H_2(W; \mathbb{Q}[\mathbb{Z}])$ is the same as the intersection (with coefficients) between $D$ and the $-1$ push-off of $\gamma$ in the cover of $M(\text{unknot})$. 

\[
\langle S, S \rangle = -t^{-1} - t - 1 + 1, \text{ so that } \left[ -t^{-1} - t - 1 + 1 \right] \text{ is the twisted intersection matrix of } W_{-1}.\]
Computing $\sigma^2(W)$:

- the curve, $\gamma$, lifts to the universal abelian cover of $M(\text{unknot})$ where it bounds an embedded disk, $D$. This disk together with the core of the 2-handle glued to $\gamma$ form the 2-sphere $S$ which generates $H_2(W; \mathbb{Q}[\mathbb{Z}])$.
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- The self intersection of $S$ in $H_2(W; \mathbb{Q}[\mathbb{Z}])$ is the same as the intersection (with coefficients) between $D$ and the $-1$ push-off of $\gamma$ in the cover of $M(\text{unknot})$.
- counting these, $\langle S, S \rangle = -t - t^{-1} + 1$, so that $[-t - t^{-1} + 1]$ is the twisted intersection matrix of $W$
Computing $\sigma^2(W)$:

The transform from $l^2(\mathbb{Z})$ to $L^2(S^1)$ ($S^1 \subseteq \mathbb{C}$) sending $t^n$ to $z^n$ is an isomorphism: $[1 - t - t^{-1}]$ has the same signature as $[1 - z - z^{-1}] = [1 - 2 \text{Re}(z)]$ which is the integral of $\text{sign}(1 - 2 \text{Re}(z))$ over $S^1$, which is $\frac{1}{3}$. Thus, $\rho^0(K) = \frac{1}{3} - (-1) = \frac{4}{3}$.
Computation for the link, $L_2$

In general this can be duplicated for any link arising as ±1 surgery along commutator curves on the unlink.
Computation for the link, $L_2$

In general this can be duplicated for any link arising as $\pm 1$ surgery along commutator curves on the unlink. With this in mind, $L_2$ can be realized as such.
Computation for the link, $L_2$

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1 & 0 & 0 & 0 \\
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\end{pmatrix},
$$
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\end{pmatrix},
$$

and twisted intersection form (computed by taking the algorithm outlined for the trefoil knot and implementing it on a computer)

$$
\begin{pmatrix}
1 + xy^2 + y + x^{-1}y^{-2} + y^{-1} & -xy - y^{-2} & xy + y^{-2} \\
-y^2 - x^{-1}y^{-1} & y + y^{-1} & -y + x^{-1} - y^{-1} \\
y^2 + x^{-1}y^{-1} & -y + x - y^{-1} & y + y^{-1} \\
\end{pmatrix}
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which have signatures 4 and \( \sim .38 \) (numerically integrating via computer).
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which have signatures 4 and $\sim .38$ (numerically integrating via computer). Thus, $\rho^0(L_2) \sim -3.62$ and $2\rho^1(K_{-7}) \lesssim -2.62$ ($-7 = -2^2 - 2 - 1$), and in particular is not zero.
Computation for the link, $L_x$

$L_{x+1}$ can be realized from $L_x$ by $+1$ surgery along $2x - 1$ new commutator curves.
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**Proposition**

*If a link $L'$ is realized as $+1$ surgery along commutator curves on another link $L$, then $\rho^0(L') \leq \rho^0(L)$*
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Thus $\rho^0(L_x) \lesssim -3.62$ for all $x \geq 2$ and $\rho^1(T_{-x^2-x-1}) \neq 0$. 
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Thus $\rho^0(L_x) \lesssim -3.62$ for all $x \geq 2$ and $\rho^1(T_{-x^2-x-1}) \neq 0$. The knots have distinct, prime Alexander polynomials and so are linearly independent.
Anisotropic knots and infection.

Definition

\( \tilde{\rho}_p^1 \) is some other localized \( \rho \)-invariant for knots.
Anisotropic knots and infection.

Theorem

Given Slice knots $R_1, \ldots, R_n$, infecting curves $\eta_1, \ldots, \eta_n$ representing elements of $A_0(R_n)$ which sit in the sum of no pair of isotropic submodules and $p$-anisotropic knots $K_1, \ldots, K_m$, if

$$n \sum_{i=1}^{n} \tilde{\rho}_1 R_i(\eta_1, (R_2(\eta_2, \ldots (R_n(\eta_n(K_i)) \ldots))))$$ is slice

then

$$\sum_{i=1}^{n} \rho_p^1(K_i) = 0$$

The following is an immediate application.
Anisotropic knots and infection.

Corollary

Let $R$ be a slice knot whose Alexander module has a unique nontrivial isotropic submodule. Let $\eta$ be an arc representing an element of $A_0(R)$ which is not isotropic. Then $\{R(\eta, R(\eta, \ldots R(\eta(K_n)) \ldots)) | n = -x^2 - x - 1, x \geq 2\}$ is linearly independent, where infection is understood to take place some fixed number of times.