$\rho$-invariants via link signatures and the linear independence of the twist knots.

Christopher William Davis,
Rice University

January 5, 2012
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Knot concordance

Definition

A **Knot** is an isotopy class of embeddings of the circle $S^1$ into the three-sphere, $S^3$. 

![Image of knots](attachment:image.png)
Knot concordance

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There is an operation called connected sum on knots. Under this operation the set of knots forms a commutative monoid.
Knot concordance.

**Definition**

A knot sitting in $S^3 = \partial B^4$ is called slice if bounds an embedded locally flat disk in $B^4$.

Modulo slice knots, knots form an abelian group under connected sum, the Knot Concordance Group, $\mathcal{C}$. 
Algebraic concordance and derivatives

In 1969, J. Levine defined a quotient of $C$, the algebraic concordance group, $AC$. A knot, $K$, bounding a genus $g$ surface $F$ in $S^3$ is called algebraically slice if there is a $g$-component nonseparating link $L = L_1, \ldots, L_g$ on $F$ such that the linking numbers $\text{lnk}(L_i, L_j)$ vanish for all $i$ and $j$. Using language of Cochran-Harvey-Leidy, the link $L$ is called a derivative of $K$. $AC := C_{\text{algebraically slice knots}}$ (J. Levine) $\sim = \mathbb{Z}_\infty \oplus \mathbb{Z}/2_\infty \oplus \mathbb{Z}/4_\infty$. 

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Algebraic concordance and derivatives

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A knot, $K$, bounding a genus $g$ surface $F$ in $S^3$ is called algebraically slice if there is a $g$-component nonseparating link $L = L_1, \ldots, L_g$ on $F$ such that the linking numbers $\text{link}(L_i, L_j^+)$ vanish for all $i$ and $j$.

Using language of Cochran-Harvey-Leidy, the link $L$ is called a derivative of $K$.

$$\mathcal{AC} := \frac{C}{\text{algebraically slice knots}}$$

(J. Levine) $\mathcal{AC} \cong \mathbb{Z}^\infty \oplus \mathbb{Z}/2^\infty \oplus \mathbb{Z}/4^\infty$
The concordance of twist knots

$T_n$ is linearly independent in $AC$.

$T_n$ is algebraically slice if and only if $n = c^2 - c$ for some $c \in \mathbb{Z}$.

$T_n$ is of order 2 in $AC$ if $n = a^2 - a + b^2$. Otherwise $T_n$ is order 4 in $AC$.

The span of the twist knots in $AC$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4$.

The 0 and 2-twist knots are slice, the 1-twist knot is order 2 in $C$.

(Casson-Gordon 1978) With those two exceptions none of the twist knots are slice.

(Jaing 1981) The twist knots which are trivial in $AC$ are linearly independent in $C$ with these two exceptions.
(J. Levine 1969) \( \{ T_n : n < 0 \} \) is linearly independent in \( AC \). \( T_n \) is algebraically slice if and only if \( n = c^2 - c \) for some \( c \in \mathbb{Z} \). \( T_n \) is of order 2 in \( AC \) if \( n = a^2 - a + b^2 \). Otherwise \( T_n \) is order 4 in \( AC \). The span of the twist knots in \( AC \) is isomorphic to \( \mathbb{Z}^\infty \oplus \mathbb{Z}/2^\infty \oplus \mathbb{Z}/4^\infty \).
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The concordance of twist knots

\begin{itemize}
  \item (J. Levine 1969) \{ T_n : n < 0 \} is linearly independent in \( AC \). \( T_n \) is algebraically slice if and only if \( n = c^2 - c \) for some \( c \in \mathbb{Z} \). \( T_n \) is of order 2 in \( AC \) if \( n = a^2 - a + b^2 \). Otherwise \( T_n \) is order 4 in \( AC \). The span of the twist knots in \( AC \) is isomorphic to \( \mathbb{Z}^\infty \oplus \mathbb{Z}/2^\infty \oplus \mathbb{Z}/4^\infty \).
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(Livingston-Naik 2001) Let $p_i$ be an enumeration of the primes congruent to 3 mod 4. Let $n_i = p_{2i-1} p_{2i} - 1$. \{ $T_{n_i}$ \} is linearly independent in the $C$. Each of these twist knots is algebraically of order 4.
The concordance of twist knots

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(S. Kim 2005) No nontrivial linear combination of the twist knots, except for the 0, 1 and 2 twist knot is ribbon.
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(S. Kim 2005) No nontrivial linear combination of the twist knots, except for the 0, 1 and 2 twist knot is ribbon.

(Lisca 2007) Except for \( n = 0, 1 \) and 2-twist knots, \( T_n \) is of infinite order in the smooth concordance group.
The concordance of twist knots

Theorem (D.)

The set containing all of the twist knots $T_n$ which are order two in $\mathcal{AC}$ is linearly independent in $\mathcal{C}$ with the following 12 possible exceptions:

Von Neumann $\rho$-invariants

For a connected, closed, oriented 3-manifold $M$ with a homomorphism, $\phi : \pi_1(M) \to G$, 

...
Von Neumann $\rho$-invariants

For a connected, closed, oriented 3-manifold $M$ with a homomorphism, $\phi : \pi_1(M) \to G$, if there is a 4-manifold $W$ with $\partial W = M$ and homomorphism $\psi : \pi_1(W) \to H$ such that there is a monomorphism $f : G \to H$ making following diagram commute:

\[
\begin{array}{ccc}
\pi_1(M) & G & \pi_1(W) \\
\downarrow & \downarrow & \downarrow \\
\phi & \downarrow & \psi \\
\downarrow & \downarrow & \downarrow \\
f & H & \\
\end{array}
\]

Then $\rho(M, G) := \sigma(W, H) - \sigma(W)$ where $\sigma(W)$ is the regular signature of the 4-manifold $W$ and $\sigma(W, H)$ is the $L^2$-signature of $W$ twisted by $\psi$.
Von Neumann $\rho$-invariants

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\[
\begin{array}{c}
\pi_1(M) \\ \downarrow \\
\pi_1(W)
\end{array}
\xrightarrow{\phi}
\begin{array}{c}
G \\ \downarrow f
\end{array}
\xrightarrow{\psi}
\begin{array}{c}
H
\end{array}
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$$
\begin{array}{c}
\pi_1(M) \xrightarrow{\phi} G \\
\downarrow \quad \downarrow f \\
\pi_1(W) \xrightarrow{\psi} H
\end{array}
$$

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$$\rho(M, G) := \sigma^{(2)}(W, H) - \sigma(W)$$
Von Neumann $\rho$-invariants

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\pi_1(M) & \xrightarrow{\phi} & G \\
\downarrow & & \downarrow f \\
\pi_1(W) & \xrightarrow{\psi} & H
\end{array}
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Then

$$\rho(M, G) := \sigma^{(2)}(W, H) - \sigma(W)$$

where $\sigma(W)$ is the regular signature of the 4-manifold $W$ and $\sigma^{(2)}(W, H)$ is the $L^2$-signature of $W$ twisted by $\psi$. 
$\rho$-invariants of zero surgery and slice knots

If $K$ is slice and $D$ is a slice disk, then $M(K)$ (zero surgery on $K$) is the boundary of $E(D)$, the compliment of $D$ in $B^4$.
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**Theorem (Cochran-Orr-Teichner)**

If $K$ is a slice knot, $D$ is a slice disk for $K$ and $\phi : \pi_1(E(D)) \to G$ is a homomorphism to a PTFA (Poly-Torsion-Free-Abelian) group then 

$$\rho(M(K), G) = \sigma^{(2)}(E(D), G) - \sigma(E(D)) = 0.$$
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If you know that a homomorphism from $\pi_1(M(K))$ must extend over a slice disk compliment if one exists then you can check its $\rho$-invariant to obstruct sliceness.
For a knot $K$, $\rho^1$ is the $\rho^1$-invariant corresponding to the quotient by double commutators.

For a group $G$, $G(1) = [G, G]$ and $G(2) = [G(1), G(1)]$. 
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$$
\rho^1(K) = \rho \left( M(K), \frac{\pi_1(M(K))}{\pi_1(M(K)(2))} \right).
$$
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For a group $G$, $G^{(1)} = [G, G]$ and $G^{(2)} = [G^{(1)}, G^{(1)}]$. 
\[ \rho^1 \text{ and concordance.} \]

**Theorem (D.)**

*If \( K_1, \ldots, K_n \) are knots with distinct prime Alexander polynomials, vanishing Tristram-Levine signatures and nonzero \( \rho^1 \)-invariants, then \( \{K_1, \ldots, K_n\} \) is linearly independent in the knot concordance group.*
$\rho^1$ and concordance.

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If $K_1, \ldots, K_n$ are knots with distinct prime Alexander polynomials, vanishing Tristram-Levine signatures and nonzero $\rho^1$-invariants, then 
\{ $K_1, \ldots, K_n$ \} is linearly independent in the knot concordance group.

\{ $T_n : n > 0$ and $n$ not of the form $c^2 - c$ \} is known to satisfy all of these conditions except for the $\rho^1$-condition.
\(\rho^1\) and concordance.

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I still need to say how to compute \(\rho^1\).
For a link $L$, $M(L)$ is the zero framed surgery along $L$. 
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**Definition**

For a link $L$ with zero pairwise linking numbers, $\rho^0(L) := \rho(M(L), \mathbb{Z}^n)$
$\rho^1$ and $\rho^0$. 
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**Theorem (D.)**

If $K$, a genus 1 knot, has prime Alexander polynomial and $K \# K$ is algebraically slice, let $L$ be a two component derivative. If additionally, the components of $L$ together with the meridians of the bands on which $L$ sit form a $\mathbb{Z}$-linearly independent set in the Alexander module of $K \# K$, then

$$|2\rho^1(K) - \rho^0(L)| \leq 1.$$
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$$|2\rho^1(K) - \rho^0(L)| \leq 1.$$ 

Thus, any tools which make $\rho^0(L)$ more computable can be used to compute $\rho^1$ and obstruct linear dependence in $C$. 
$\rho^0$ as an integral

Cimsoni and Florens associate to an $m$-colored link $L$ (A link whose components are decorated with an integer $1, \ldots, m$) a signature function, $\sigma_L : T^m \rightarrow \mathbb{Z}$.

$(T^n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z_k| = 1\})$
\( \rho^0 \) as an integral

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**Theorem (D.)**

For an \( n \)-component link \( L \) with zero pairwise linking numbers,

\[ \rho^0(L) = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} \sigma_L(\omega) d\omega, \]

where \( \sigma_L \) is the Cimsoni-Florens signature function, regarding \( L \) as an \( n \)-colored link.
\( \rho^0 \) as an integral

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**Notation:** Even if \( L \) has nonzero linking numbers or multiple components of the same color I will refer to this integral as \( R(L) \).
Properties of the Cimasoni-Florens signature

For knots the Cimasoni-Florens signature agrees with the Tristram-Levine signature. For 1-colored links it agrees with the Murasugi signature. \( \sigma \)

\( \sigma \) \((\omega)\) adds under split union. Some moves change \( \sigma \) by \(0, 1, \) or \(-1\):

- smoothing a one colored crossing,
- changing a two colored crossing and adding a band between like colored arcs.

(Cimasoni-Florens say exactly how it changes the signature.)

Adding a band between two split sublinks does not change \( \sigma \).
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Properties of the Cimasoni-Florens signature

- For knots the Cimasoni-Florens signature agrees with the Tristram-Levine signature. For 1-colored links it agrees with the Murasugi signature.
- $\sigma_L(\omega)$ adds under split union
- Some moves change $\sigma_L(\omega)$ by 0, 1, or $-1$: smoothing a one colored crossing,

\[ \xymatrix{ \begin{array}{c} \includegraphics[width=2cm]{crossing1.png} \end{array} \end{array} \begin{array}{c} \xrightarrow{\text{smooth}} \end{array} \begin{array}{c} \includegraphics[width=2cm]{crossing2.png} \end{array} \]
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- Some moves change $\sigma_L(\omega)$ by 0, 1, or $-1$: smoothing a one colored crossing, changing a two colored crossing.
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- $\sigma_L(\omega)$ adds under split union
- Some moves change $\sigma_L(\omega)$ by 0, 1, or $-1$: smoothing a one colored crossing, changing a two colored crossing and adding a band between like colored arcs.

![Diagram](image-url)
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- Some moves change $\sigma_L(\omega)$ by 0, 1, or $-1$: smoothing a one colored crossing, changing a two colored crossing and adding a band between like colored arcs. (Cimasoni-Florens say exactly how it changes the signature.)

- Adding a band between two split sublinks does not change $\sigma_L(\omega)$. 

\[\begin{align*}
\text{Crossing smoothing} & \rightarrow \text{Smoothing a one colored crossing} \\
\text{Two colored crossing} & \rightarrow \text{Changing a two colored crossing} \\
\text{Adding a band} & \rightarrow \text{Adding a band between like colored arcs}
\end{align*}\]
Integrating these facts

- For knots the $R(K)$ is the integral of the Tristram-Levine signature. For 1-colored links it is the integral of the Murasugi signature.
- $R(L)$ adds under split union
- Some moves change $R(L)$ by a real number in $[-1, 1]$: smoothing a one colored crossing, changing a two colored crossing and adding a band between like colored arcs.
- Adding a band between two split components does not change $R(L)$. 

\[\text{Diagram with examples of moves and their effects.}\]
One more move

\[ a \text{ strands} \quad \leftrightarrow \quad b \text{ strands} \quad \Rightarrow \quad B \]
One more move

Proposition (D.)

\( R(L) \) and \( R(L') \) differ by at most \( a + b - 1 \)
Application: Twist knots

Lemma

$T_n$ is algebraically of order 2 if and only if $n = a^2 - a + b^2$ for positive integers $a$ and $b$ but $n$ is not given by $c^2 - c$ for a positive integers $c$. 
Application: Twist knots

Lemma

$T_n$ is algebraically of order 2 if and only if $n = a^2 - a + b^2$ for positive integers $a$ and $b$ but $n$ is not given by $c^2 - c$ for a positive integers $c$. The link $L_{a,b}$ is a derivative for $T_n \# T_n$. 

Figure: For $n = a^2 - a + b^2$, the link $L_{a,b}$ is a derivative for $T_n \# T_n$. The curves $m_1$ and $m_2$ are meridians about the bands on which the components of $L_{a,b}$ sit.
Application: Twist knots

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$b$ strands of each color

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Application: Twist knots

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$T_n$ is algebraically of order 2 if and only if $n = a^2 - a + b^2$ for positive integers $a$ and $b$ but $n$ is not given by $c^2 - c$ for a positive integers $c$. The link $L_{a,b}$ is a derivative for $T_n \# T_n$.

![Diagram of a twist knot with $n$ strands of each color and $a - b$ strands](image)
Lemma

$T_n$ is algebraically of order 2 if and only if $n = a^2 - a + b^2$ for positive integers $a$ and $b$ but $n$ is not given by $c^2 - c$ for a positive integers $c$. The link $L_{a,b}$ is a derivative for $T_n \# T_n$. 

![Diagram of a twist knot with labels for strands and components.](Image)
Application: Twist knots

Lemma

$T_n$ is algebraically of order 2 if and only if $n = a^2 - a + b^2$ for positive integers $a$ and $b$ but $n$ is not given by $c^2 - c$ for a positive integers $c$. The link $L_{a,b}$ is a derivative for $T_n \# T_n$.

Observe then $\{ T_n : n = a^2 - a + b^2 \text{ and } |\rho^0(L_{a,b})| > 1 \}$ is linearly independent.
Simplify $L_{a,b}$

Figure: $T(4,3)$, the $(4, -3)$-torus knot.
Simplify $L_{a,b}$

\[ L_{a,b} \rightarrow \text{Add blue and red bands.} \]

\[ \Rightarrow \text{Change two-color crossings.} \]

\[ \Rightarrow \text{Erase some B's.} \]

\[ L''''' = T(a,1-a) \sqcup T(1-a,a) \sqcup T(b,1-b) \sqcup T(b,1-b) \] is a split union of one-colored torus links.

Figure: $T(4,3)$, the $(4,3)$-torus knot.
Simplify $L_{a,b}$

Add $b$ red and $b$ blue bands.

$R(L_{a,b})$ is within $2b - 1$ of $R(L'_{a,b})$
Simplify $L_{a,b}$

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$R(L_{a,b})$ is within $3b - 1$ of $R(L''_{a,b})$
Simplify $L_{a,b}$

$R(L_{a,b})$ is within $3b - 1$ of $R(L''_{a,b})$

change $b$ two-color crossings.
Simplify $L_{a,b}$

\[ L_{a,b} \rightarrow \]

Erase some $B$'s.

$R(L_{a,b})$ is within $3b - 1 + (a + b - 1) + ((a - 1) + b - 1)$ of $R(L_{a,b}^{\prime\prime\prime})$
Simplify $L_{a,b}$

$R(L_{a,b})$ is within $3b - 1 + (a + b - 1) + ((a - 1) + b - 1)$ of $R(L''')$

$L'''_{a,b} = T(a, 1 - a) \sqcup T(1 - a, a) \sqcup T(b, -b) \sqcup T(b, -b)$ is a split union of one-colored torus links.
Simplify $L_{a,b}$

Add $b$ red and $b$ blue bands.

$\rightarrow$ change $b$ two-color crossings.

$\rightarrow$ Erase some $B$’s.

$R(L_{a,b})$ is within $3b - 1 + (a + b - 1) + ((a - 1) + b - 1)$ of $R(L''')$

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Figure: $T(4, 3)$, the $(4, -3)$-torus knot.
Put it all together: bounding $\rho^0(L_{a,b})$.

Thus,

$$\rho^0(L_{a,b}) \geq R(T(a, 1-a)) + R(T(1-a, a)) + 2R(T(b, -b)) - (2a + 5b - 4)$$

where $T(x, y)$ is the $(x, y)$-torus link.
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Collins (2010) and Borodzik (2010) compute the integral of the Tristram-Levine signature of torus knots:
Put it all together: bounding $\rho^0(L_{a,b})$.

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Collins (2010) and Borodzik (2010) compute the integral of the Tristram-Levine signature of torus knots:
$$R(T(a, 1-a)) = R(T(1-a, a)) = \frac{(a + 1)(a - 2)}{3}.$$
Put it all together: bounding $\rho^0(L_{a,b})$.

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Collins (2010) and Borodzik (2010) compute the integral of the Tristram-Levine signature of torus knots:
$$R(T(a, 1-a)) = R(T(1-a, a)) = \frac{(a + 1)(a - 2)}{3}.$$ 

According to Borodzik, $R(T(b, -b)) = \frac{b^2 - 1}{3}$. 

Putting all this together

Thus, $\rho^0(L_{a,b}) = R(L_{a,b}) \geq \frac{2a^2 + 2b^2 - 8a - 19b + 10}{3}$. This is greater than 1 for all but finitely many positive integers $a$ and $b$. 

Theorem (D.)
Putting all this together

Thus, \( \rho^0(L_{a,b}) = R(L_{a,b}) \geq \frac{2a^2 + 2b^2 - 8a - 19b + 10}{3} \). This is greater than 1 for all but finitely many positive integers \( a \) and \( b \).
Putting all this together

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**Theorem (D.)**

The set containing all twist knots $T_n$ of algebraic order two, with only finitely many exceptions is linearly independent.
Putting all this together

Thus, $\rho^0(L_{a,b}) = R(L_{a,b}) \geq \frac{2a^2 + 2b^2 - 8a - 19b + 10}{3}$. This is greater than 1 for all but finitely many positive integers $a$ and $b$.

**Theorem (D.)**

The set containing all twist knots $T_n$ of algebraic order two, with 39 possible exceptions:

$$n = 1, 3, 4, 9, 10, 11, 15, 16, 18, 22, 24, 25, 27, 28, 29, 34, 36, 37, 38, 39, 45, 48, 49, 51, 55, 58, 61, 64, 66, 67, 69, 70, 78, 79, 83, 84, 87, 93, 101.$$
Pair this result with work of Tamulais, saying that \( \{ T_n : n \geq 3, 4n + 1 \text{ is prime} \} \) is linearly independent.
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\( \{ T_n : n \geq 3, 4n + 1 \text{ is prime} \} \) is linearly independent.

**Theorem (D.)**

The set containing all of the twist knots \( T_n \) which are algebraically of order two is linearly independent with the following 12 possible exceptions:

\[
\]
Thanks!