Math 316, Intro to Analysis.

Groupwork: The Nested intervals theorem and the uncountability of the real numbers

You may have heard the fact that \( \mathbb{R} \) is not countable. Today we prove it.

Remark. The book explores a different proof of this fact. It uses the fact that real numbers between 0 and 1 can be represented as decimals (things like .20183450204... ) while somehow satisfying, the proof is incomplete without an argument that real numbers are the same as decimals. This document will guide you through an argument using the fact (which you will prove) that an infinite intersection of nested closed intervals is not empty.

1. The Nested Intervals Theorem

We begin by proving the seemingly unrelated nested intervals theorem (Theorem 6). Recall that for real numbers \( a < b \), the closed interval from \( a \) to \( b \) is defined by \( [a, b] = \) and the open interval from \( a \) to be is defined by \( (a, b) = \)

Prove the following lemma as a warm up.

Lemma 1. \( \bigcap_{n=1}^{\infty} [0, \frac{1}{n}] = \{0\} \).

Recall the formal definition of the infinite intersection. For sets \( A_1, A_2, \ldots \), the infinite intersection is given by \( \bigcap_{n=1}^{\infty} A_n = \{a \text{ such that for all } n \in \mathbb{N}, \ a \in A_n\} \). You might make use of the fact that \( \inf \{1/n : \ n \in \mathbb{N}\} = 0 \).

In particular notice that \( \bigcap_{n=1}^{\infty} (0, \frac{1}{n}) \) is empty (Take a moment and convince yourself of this.) while \( \bigcap_{n=1}^{\infty} [0, \frac{1}{n}] \) is not. The nested intervals theorem says that if you take any sequence of nested closed intervals intervals of the real line, their infinite intersection will be non-empty.
Definition 2. Nested Intervals: a sequence of intervals $I_1, I_2, \ldots$ is called nested if $I_m \supseteq I_n$ whenever $n \leq m$.

Prove the following equivalences.

Theorem 3. Let $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ with $I_n = [a_n, b_n]$ be a sequence of nested intervals. Let $x \in \mathbb{R}$. Then $x \in \bigcap_{n=1}^{\infty} [a_n, b_n]$ if and only if $x$ is an upper bound for $\{a_1, a_2, \cdots\}$ and is a lower bound for $\{b_1, b_2, \cdots\}$.

Proof. Let $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ with $I_n = [a_n, b_n]$ be a sequence of nested intervals. Let $x \in \mathbb{R}$.

$(\implies)$ Suppose that $x \in \bigcap_{n=1}^{\infty} [a_n, b_n]$. In order to prove that $x$ is an upper bound for $\{a_1, a_2, \cdots\}$ and is a lower bound for $\{b_1, b_2, \cdots\}$ we must prove the inequality:

\[ a_n \leq x \leq b_n \]

for every $n \in \mathbb{N}$.

Since $x \in \bigcap_{n=1}^{\infty} [a_n, b_n]$ the definition of the infinite intersection implies that $x \in [a_n, b_n]$ for all $n \in \mathbb{N}$. So that by the definition of the interval $I_n = [a_n, b_n]$ we get the inequality

\[ a_n \leq x \leq b_n \]

for every $n \in \mathbb{N}$, and we conclude that $x$ is an upper bound for $\{a_1, a_2, \cdots\}$ and is a lower bound for $\{b_1, b_2, \cdots\}$.

$(\impliedby)$ Conversely, suppose that $x$ is an upper bound for $\{a_1, a_2, \cdots\}$ and is a lower bound for $\{b_1, b_2, \cdots\}$. Then for all $n \in \mathbb{N}$, we get the inequality $a_n \leq x \leq b_n$. Then by the definition of an interval we see that $x \in [a_n, b_n]$ for every $n \in \mathbb{N}$. By the definition of the infinite intersection we conclude that $x \in \bigcap_{n=1}^{\infty} [a_n, b_n]$, completing the proof.

Theorem 3 shows that finding an element of $I_1 \cap I_2 \cap \cdots$, (which we need if we are to show that the intersection is not empty) is equivalent to finding an upper bound for $\{a_1, a_2, \cdots\}$ which is also a lower bound for $\{b_1, b_2, \cdots\}$. Let’s show that there exist upper and lower bounds.
Proposition 4. Let $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ with $I_n = [a_n, b_n]$ be a sequence of nested intervals. For every $j, k \in \mathbb{N}$, $a_j \leq b_k$.

Proof. Let $j, k \in \mathbb{N}$. Case 1: If $j = k$ then the amounts to showing that $a_j \leq b_j$. Since $[a_j, b_j] = I_j$ is an interval, what inequality MUST $a_j$ and $b_j$ satisfy? (Glance at the first page where intervals are defined.)

Case 2: If $k < j$ then by assumption $[a_j, b_j] = I_j$ is a subset of $[a_k, b_k] = I_k$. Prove that $a_j \in I_k$.

Why does this imply that $a_j \leq b_k$?

Case 2: If $j < k$ By assumption $[a_j, b_j]$ ________ $[a_k, b_k]$. Which is true: $b_j \in I_k$ or $b_k \in I_j$? Prove it

Why does this imply that $a_j \leq b_k$?

Look at the proposition you just proved. It says that every $b_k$ is (an upper / a lower) ________ bound for $\{a_n : n \in \mathbb{N}\}$ and that every $a_j$ is (an upper / a lower) ________ bound for $\{b_n : n \in \mathbb{N}\}$.

What does the completeness of $\mathbb{R}$ say about these sets?

Theorem 5. Let $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ with $I_n = [a_n, b_n]$ be a sequence of nested intervals.

Then $\sup(\{a_k : k \in \mathbb{N}\})$ is in $\bigcap_{n=1}^{\infty} I_n$.

Alternately, you may instead prove that $\inf\{b_1, b_2, \cdots\}$ is in the intersection.

Proof. Let $s$ be the supremum of $\{a_k : k \in \mathbb{N}\}$. We want to show that $x \in \bigcap_{n=1}^{\infty} I_n$. Thanks to Theorem 3, this is equivalent to showing that:
\[ s \text{ is an upper bound on } \{a_i\} \]
\[ s \text{ is a lower bound on } \{b_i\} \]

Let's start with the first of these conditions. **Claim:** \( s \) is an upper bound for \( \{a_k : k \in \mathbb{N}\} \). (This one is easy.)

Now the second. **Claim:** \( s \) ia a lower bound for \( \{b_k : k \in \mathbb{N}\} \)?
Consider any \( b_k \in \{b_k : k \in \mathbb{N}\} \), According to the remark just following Theorem 4

Notice that we have found an element of \( I_1 \cap I_2 \cap \ldots \). Sets which have at least one element are not empty:

**Corollary 6** (The nested intervals theorem). Let \( I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots \) with \( I_n = [a_n, b_n] \) be a sequence of nested intervals. Then \( I_1 \cap I_2 \cap \cdots \neq \emptyset \).

2. **The Uncountability of the Reals**

We can now prove our stated main result.

**Theorem 7.** There does not exist a surjection from \( \mathbb{N} \) to \( \mathbb{R} \).

In the language of cardinality from set theory, this means that **The real numbers are uncountably infinite.** Let's prove Theorem 7. That is, for every \( f : \mathbb{N} \to \mathbb{R} \) we need to find a number which \( f \) misses.

Start with the following Lemma.

**Lemma 8.** For any closed interval \( I = [a, b] \) and any \( x \in \mathbb{R} \), there exists a closed interval \( I' \) with \( I' \subseteq I \) and \( x \notin I' \).

**HINT:** It may be helpful to break this up into the cases (1) The silly case: \( x \notin I \) (2) the boundary case: \( x = a \) or \( x = b \) (3) the interior case: \( a < x < b \). In each case consider drawing a picture to motivate a choice of \( I' \).

Case (1) Suppose that \( x \notin I \). Take \( I' = \). Prove that \( x \notin I' \).
Case (2a) Suppose that $x = a$. Let $I' =$ . Prove that $x \notin I'$

Case (2b) Suppose that $x = b$. Let $I' =$ . Prove that $x \notin I'$

Case (3) Suppose that $a < x < b$. Let $I' =$ . Prove that $x \notin I'$

Using Lemma 8, perform the following construction needed to prove the following result

**Proposition 9.** Let $x_1, x_2, \cdots$ be any sequence of real numbers. There is a sequence of nested closed intervals $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ such that for all $k \in \mathbb{N}$ $x_k \notin I_k$.

**Proof.** (Base Step)
Let $x_1 \in \mathbb{R}$. We need a closed interval $I_1$ such that $x_1 \notin I_1$. Take $I_1 =$ (perhaps a picture will help you choose $I_1$). Prove that $x_1 \notin I_1$

(Inductive step) Consider any natural number $n \in \mathbb{N}$. Suppose that there are intervals $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n$ such that $x_k \notin I_k$ for $k = 1, 2, \ldots n$. According to Lemma 8 there exists an interval $I_{k+1}$ (called $I'$ in Lemma 8) such that:
Explain why $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq I_{n+1}$ and that $x_{n+1} \notin I_{n+1}$.

By iterating this procedure, we construct a sequence of nested closed intervals $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ with $x_k \notin I_k$ for all $k$. This completes the proof. □

For the sequence of nested intervals $(I_k)$ produced by Proposition 9, Corollary 6 implies that $I_1 \cap I_2 \cap \cdots$ is not empty. Interesting. Let’s find out if elements of $I_1 \cap I_2 \cap \cdots$ can be hit by the sequence.

**Proposition 10.** Let $x_1, x_2, \cdots$ be any sequence. Let $I_1 \supseteq I_2 \supseteq \cdots$ with $x_k \notin I_k$ be a sequence of nested intervals with $x_k \notin I_k$ for all $k$. Let $y \in I_1 \cap I_2 \cap \cdots$. It follows that $x_k \neq y$ for all $k$.

Say a few sentences proving Theorem 7

**Proof of Theorem 7.** Let $f : \mathbb{N} \rightarrow \mathbb{R}$ be any function. We must prove that $f$ is not onto. Let $x_n$ be the sequence defined by $x_n = f(n)$. □