The uncountability of the real numbers

You may have heard the fact that \( \mathbb{R} \) is not countable. Today we make sense of this fact and
prove it. Your goal is the proof of the following theorem:

**Theorem 1.** if \( f \) is any map from \( \mathbb{N} \) to \( \mathbb{R} \) then there is a real number not in the image of \( f \).

Another way to parse this theorem is by saying that if you start trying to list all the real numbers
as \( x_1, x_2, x_3, \ldots \) then even if you continue on forever then you will miss at least one real number.

**Remark:** The book explores a different proof of this fact. It uses the fact that real numbers
between 0 and 1 can be represented as decimals (things like \( .20183450204 \ldots \)) while somehow
satisfying, the proof is incomplete without an argument that real numbers are the same as decimals.
This document will guide you through an argument using the fact (which you will prove) that an
infinite intersection of closed intervals is not empty.

1. **The Nested Intervals Theorem**

We begin by proving the seemingly unrelated **nested intervals theorem** (Corollary ??). Recall
that for real numbers \( a < b \), the closed interval from \( a \) to \( b \) is defined by \( [a, b] = \)

and the open interval from \( a \) to be is defined by \( (a, b) = \)

**Warm Up 2.** Show that \( [0, 1] \cap [0, 1/2] \cap [0, 1/3] \cap \cdots = \{0\} \) and that \( (0, 1) \cap (0, 1/2) \cap (0, 1/3) \cap \cdots = \{\} = \emptyset \)

**Definition 3.** **Nested Intervals:** a sequence of intervals \( I_1, I_2, \cdots \) is called nested if

The fact that an infinite intersection of nested open intervals can be empty while (this partic-
ular) infinite intersection of nested closed intervals is not empty is suggestive. Maybe it is true
in general that an infinite intersection of nested closed intervals is non-empty.
Prove the following equivalences.

**Theorem 4.** Let $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ with $I_n = [a_n, b_n]$ be a sequence of nested intervals. Let $x \in \mathbb{R}$.

Show that following are equivalent:

1. $x \in I_1 \cap I_2 \cap \cdots$
2. $x$ is in $[a_n, b_n]$ for all $n$.
3. For all $n$, $a_n \leq x \leq b_n$.
4. $x$ is an upper bound for $\{a_1, a_2, \cdots\}$ and is a lower bound for $\{b_1, b_2, \cdots\}$

**Proof.** (1) $\iff$ (2)

Suppose that $x \in I_1 \cap I_2 \cap \cdots$

So that $x$ is in $[a_n, b_n]$ for all $n$.

Conversely suppose that $x$ is in $[a_n, b_n]$ for all $n$.

(2) $\iff$ (3)

Suppose that $x$ is in $[a_n, b_n]$ for all $n$.

So that $x \in I_1 \cap I_2 \cap \cdots$.

(3) $\iff$ (4)

Suppose that for all $n$, $a_n \leq x \leq b_n$.

So that for all $n$, $a_n \leq x \leq b_n$.

Conversely suppose that for all $n$, $a_n \leq x \leq b_n$. 
So that $x$ is in $[a_n, b_n]$ for all $n$.

$\text{(3)} \iff \text{(4)}$
Suppose that for all $n$, $a_n \leq x \leq b_n$.

So that $x$ is an upper bound for $\{a_n | n \in \mathbb{N}\}$ and a lower bound for $\{b_n | n \in \mathbb{N}\}$.
Suppose that $x$ is an upper bound for $\{a_n | n \in \mathbb{N}\}$ and a lower bound for $\{b_n | n \in \mathbb{N}\}$.

So that for all $n$, $a_n \leq x \leq b_n$.

\[\square\]

Theorem ?? $(1) \iff (4)$ shows that finding an element of $I_1 \cap I_2 \cap \cdots$, (which we need if we are to show that the intersection is not empty) is equivalent to finding an upper bound for $\{a_1, a_2, \cdots\}$ which is also a lower bound for $\{b_1, b_2, \cdots\}$. Let’s show that there are upper and lower bounds.

**Theorem 5.** Let $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ with $I_n = [a_n, b_n]$ be a sequence of nested intervals. Show that for every $j, k \in \mathbb{N}$, $a_j \leq b_k$.

Let $j, k \in \mathbb{N}$. Case 1: If $j = k$ then the amounts to showing that $a_j \leq b_j$. Since $[a_j, b_j] = I_j$ is an interval, what inequality MUST $a_j$ and $b_j$ satisfy? (Glance at the first page where intervals are defined.)

Case 2: If $k < j$ By assumption $[a_j, b_j] = I_j$ is a subset of $[a_k, b_k] = I_k$. Prove that $a_j \in I_k$.

Why does this imply that $a_j \leq b_k$?

Case 2: If $j < k$ By assumption $[a_j, b_j] \subseteq [a_k, b_k]$. Which is true: $b_j \in I_k$ or $b_k \in I_j$? Prove it
Why does this imply that $a_j \leq b_k$?

Look at the theorem you just proved. Find an upper bound for $\{a_j : j \in \mathbb{N}\}$: _____
Find an lower bound for $\{b_j : j \in \mathbb{N}\}$: _____
What does the completeness of $\mathbb{R}$ say about these sets?

---

**Theorem 6.** Let $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ with $I_n = [a_n, b_n]$ be a sequence of nested intervals.

Then $\sup(\{a_k : k \in \mathbb{N}\})$ is in $I_1 \cap I_2 \cap \cdots$.

If you wish to be contrary, you may instead prove that $\inf\{b_1, b_2, \cdots\}$ is in the intersection.

**Proof.** Let $s$ be the supremum of $\{a_k : k \in \mathbb{N}\}$. Thanks to Theorem ?? showing that $s$ is in the infinite intersection is equivalent to saying that

---

Lets start with the first of these conditions: Why should $s$ be an upper bound for $\{a_k : k \in \mathbb{N}\}$?

---

Now the second: Why should $s$ be an lower bound for $\{b_k : k \in \mathbb{N}\}$? **hint:** You might need Theorem ??

Consider any $b_k \in \{b_k : k \in \mathbb{N}\}$, According to Theorem ?? . . .

---

Notice that we have found an element of $I_1 \cap I_2 \cap \cdots$. Sets which have at least one element are not empty:

**Corollary 7** (The nested intervals theorem). Let $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ with $I_n = [a_n, b_n]$ be a sequence of nested intervals. Then $I_1 \cap I_2 \cap \cdots \neq \emptyset$. 

□
2. The Uncountability of the Reals

Let’s prove Theorem ???. That is, for every \( f : \mathbb{N} \to \mathbb{R} \) we need to find a number which \( f \) misses.

Prove the following Lemma.

**Lemma 8.** For any closed interval \( I = [a, b] \) and any \( x \in \mathbb{R} \), show that there is a closed interval \( I' \subseteq I \) and \( x \notin I' \).

**HINT:** It may be helpful to break this up into the cases (1) The silly case: \( x \notin I \) (2) the boundary case: \( x = a \) or \( x = b \) (3) the interior case: \( a < x < b \).

Case (1) Suppose that \( x \notin I \). Take \( I' = \) ____

_________________________________________________________________

Case (2a) Suppose that \( x = a \). Let \( a' = \) ____ and \( I' = \) ____

_________________________________________________________________

Case (2b) Suppose that \( x = b \). Let \( b' = \) ____ and \( I' = \) ____

_________________________________________________________________

Case (3) Suppose that \( a < x < b \). Let \( a' = \) ____ and \( I' = \) ____

_________________________________________________________________
Using Lemma ??, perform the following construction. Induction may come in handy.

**Theorem 9.** Let $x_1, x_2, \cdots$ be any sequence of real numbers. Show that there is a sequence of nested closed intervals $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ with $x_k \notin I_k$ for all $k$.

**Proof.** We will proceed using a construction which will feel very inductive

(Base Step)

Let $x_1 \in \mathbb{R}$. Prove that there is a closed interval $I_1$ such that $x_1 \notin I_1$. Take $I_1 = \ldots$

(Inductive step) Consider any natural number $N$. Suppose that there are intervals $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n$ such that $x_k \notin I_k$ for $k = 1, 2, \ldots n$.

Use Lemma ?? to find $I_{n+1}$.

Show that $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq I_{n+1}$ and that $x_{n+1} \notin I_{n+1}$.

For the interval $I_k$ of the above construction, Corollary ?? implies that $I_1 \cap I_2 \cap \cdots$ is not empty. Interesting. Let’s find out if elements of $I_1 \cap I_2 \cap \cdots$ can be hit by the sequence.

**Theorem 10.** Let $x_1, x_2, \cdots$ be any sequence. By Construction ?? there are closed intervals $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_k$ with $x_k \notin I_k$ for $k = 1, 2, \cdots n$. By the nested intervals theorem, there is a $y \in I_1 \cap I_2 \cap \cdots$. Prove that $x_k \neq y$ for all $k$.

Say a few sentences proving the following.

**Corollary 11.** There is no surjection from $\mathbb{N}$ to $\mathbb{R}$

Here is a good question.

We have proven that the power set of the natural numbers, $\mathcal{P}(\mathbb{N})$ and $\mathbb{R}$ are both uncountably infinite. What do you think, are they the same size? I will give some extra credit points if people can exhibit one-to-one and / or onto maps between $\mathbb{R}$ and $\mathcal{P}(\mathbb{N})$