I have added some prompts in caps. These should not appear in your final writeup. Instead replace them with some actual analysis. Feel free to also correct any typos you find.

You have proven that if you improve the concept of convergence to uniform convergence, then the limit of a sequence of continuous functions is continuous. This project will have you prove that instead of powering up the notion of convergence, you can improve the notion of continuity and get the limit to be continuous.

Definition 1. A sequence of functions \((f_n : U \to \mathbb{R})\) is called **equicontinuous** if for all \(\epsilon > 0\) and all \(x \in U\) there is a \(\delta > 0\) such that for all \(n \in \mathbb{N}\) and all \(y \in U\) if \(|x - y| < \delta\) then \(|f_n(x) - f_n(y)| < \epsilon\).

Basically, this definition says that we may allow \(\delta\) to depend on \(x\), but \(\delta\) cannot depend on \(n\). The same choice of \(\delta\) must be used for each \(f_n\), independent of \(n\).

Claim 2. Let \((f_n : \mathbb{R} \to \mathbb{R})\) be given by

\[
    f_n(x) = \begin{cases} 
        0 & \text{if } x < n \\
        x - n & \text{if } n \leq x 
    \end{cases}
\]

Then \((f_n)\) is equicontinuous on \(\mathbb{R}\).

Proof. Your proof goes here. Your proof might start with “consider any \(\epsilon > 0\) and any \(x \in U\). Let \(\delta = \ldots\) and consider any \(n \in \mathbb{N}\).” \(\square\)

The whole point of this packet is that everything that happens for uniformly convergent sequences of continuous functions holds for **pointwise** convergent **equicontinuous** sequences of functions.

Theorem 3. Let \((f_n : U \to \mathbb{R})\) be an equicontinuous sequence of functions. Let \(f : U \to \mathbb{R}\) be its pointwise limit. Then \(f : U \to \mathbb{R}\) is continuous.

Proof. Your proof goes here. Think about how we proved that the uniform limit of a sequence of continuous functions is continuous. \(\square\)

We can add the prefix “equi-” to other notions of continuity:

Definition 4. A sequence of functions \((f_n : U \to \mathbb{R})\) is called **uniformly equicontinuous** if for all \(\epsilon > 0\) there is a \(\delta > 0\) such that for all \(n \in \mathbb{N}\) and all \(x, y \in U\) if \(|x - y| < \delta\) then \(|f_n(x) - f_n(y)| < \epsilon\).

Importantly, we get good results out of this:

Theorem 5. Let \((f_n : U \to \mathbb{R})\) be an uniformly equicontinuous sequence of functions. Let \(f : U \to \mathbb{R}\) be its pointwise limit. Then \(f\) is uniformly continuous.

Proof. Your proof goes here. Make sure that your delta depends only on epsilon. \(\square\)
An important payoff of the fact that uniform convergence preserves continuity was the continuity of power series. The notion of equicontinuity also recovers the continuity of power series.

**Theorem 6.** Let $a_0, a_1, a_2, \ldots$ be a sequence of real numbers. Let
\[ \lambda = \limsup_{k \to \infty} |a_k|^{1/k} \]
and
\[ R = \begin{cases} 
\frac{1}{\lambda} & \text{if } 0 < \lambda < \infty \\
\infty & \text{if } \lambda = 0 \\
0 & \text{if } \lambda = \infty 
\end{cases} \]

(1) Then for every $L$ with $0 < L < R$, the sequence of partial sums $S_n(x) = \sum_{k=0}^{n} a_k \cdot x^k$ is uniformly equicontinuous on $[-L, L]$,

(2) $\sum_{k=0}^{\infty} a_k \cdot x^k$ diverges if $|x| > R$,

(3) The sequence of partial sum $S_n(x)$ converges pointwise on $(-R, R)$.

Using the preceding theorems on this page, we may prove the following corollary:

**Corollary 7.** Let $S(x) = \sum_{k=0}^{\infty} a_k \cdot x^k$ be a power series with radius of convergence $R$.

- If $0 < R < \infty$ then $(-R, R) \subseteq \text{Dom}(f) \subseteq [-R, R]$ and $f$ is continuous on $(-R, R)$.
- If $R = \infty$ then $S$ has domain $\mathbb{R}$ and $S$ is continuous on all of $\mathbb{R}$.
- If $R = 0$ then $\sum_{k=0}^{\infty} a_k \cdot x^k$ has domain $\{0\}$ and $f(0) = a_0$ on that domain.

**Proof of Corollary 7.** YOUR PROOF GOES HERE CONSIDER LOOKING AT THE NOTES ON POWER SERIES FOR INSPIRATION. □

**Proof of Theorem 6.** We begin our proof of Claim (1) with the following lemma.

**Lemma 8.** Let $(a_k)$ be a sequence of real numbers. Let $\lambda = \limsup_{k \to \infty} a_k^{1/k}$. Then for every number $L$ with $0 < L < \frac{1}{\lambda}$,
\[ \sum_{k=0}^{\infty} |a_k| \cdot k \cdot L^{k-1} \]
converges.

**Proof of Lemma 8.** YOUR PROOF GOES HERE CONSIDER TAKING ADVANTAGE OF THE ROOT TEST, JUST AS WE DID IN THE NOTE PACKET ABOUT POWER SERIES. □

We now prove Claim (1) of Theorem 6.

YOUR PROOF GOES HERE, MAYBE TAKE ADVANTAGE OF LEMMA 8, IF YOU DON’T USE THE LEMMA, THEN YOU DON’T NEED TO INCLUDE IT IN YOUR WRITEUP.

Next we prove Claim (2) of Theorem 6.

YOUR PROOF GOES HERE. THIS ONE IS EASIER. NOTICE THAT THIS CLAIM IS IDENTICAL TO ONE MADE IN THE NOTES ON POWER SERIES.
Finally we prove Claim (3) of Theorem 6.

YOUR PROOF GOES HERE. THIS ONE IS EASIER THAN THE ANALOGOUS RESULT FROM THE NOTES ON POWER SERIES, AS YOU ONLY NEED POINTWISE CONVERGENCE. □